Some results on sensitivity modeling of market risk via Lévy models

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Abstract

A significant portion of risk exposure of internationally active financial institutions is created by currencies, ie. positions exposed to foreign exchange (FX) rates. Although FX rate market can be regarded as a very efficient one, arrival of extraordinary information results into huge movements, usually approximated by measures of kurtosis and skewness. A very challenging family of processes, allowing us to capture these features is a so called Lévy family of processes. In this paper we select two subordinated Lévy models and look more closely at the sensitivity of capital requirement/economic capital to the input parameters, ie. kurtosis, skewness, standard deviation and degrees of freedom, of the model when the Student copula is assumed as a tool to depict the dependency among particular risk drivers of the portfolio. We found out several interesting results, mainly that skewness play almost no role within the models assumed here, as opposed to, e.g. variance.

Keywords

Capital adequacy, sensitivity. FX rates, Lévy models

1 Introduction

Risk management is a very important and essential issue within managing of financial institutions. An efficient management of financial risk can increase the performance of any given entity. However, in order to allow us to maximize the effect of risk management, we should use proper tools.

Since the recent evolution at financial markets even stress further the insufficiency of models based on Gaussian distribution, the need of models capturing both the non-normality of marginal distributions and the non-linearity in dependency among them increases – see e.g. Cont and Tankov (2004), Nelsen (2006) and references therein.

In this paper we look in particular at the sensitivity of both, the supervisors' and equity holders' indicators – the capital adequacy ratio and economic capital ratio.

We proceed as follows. Section one is devoted to a general framework of riskmanagement of financial institutions, while in Section 3 we define the marginal distribution and copula functions models. Next, we describe the data set. The results are discussed in Section 4. Section ř concludes the paper.

2 Risk assessment in financial institutions

The two most important incentives for managing the risk of financial institutions are (i) *the supervisor's requirements* and (ii) *the equity holders' interest*. While supervisors

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The research is due to the support provided by GAČR (Czech Science Foundation – Grantová Agentura České Republiky) under the project No. 402/08/1237. All support is greatly acknowledged.

formulate the rules, which must be followed by any entity desiring to do a business in a given industry, the equity holders have an inherent right to determine the policy as the owners.

2.1 Supervisors approach

The core motive of supervisors' activities in financial industry is to keep it as healthy as possible. The supervisors policy affects riskmanagement activities of financial institutions in two important ways: (i) first, they specify eligible approaches to risk measuring; (ii) second, they set risk limits, which should not be broken. Starting with The Amendment 1992, the financial institutions are eligible to use *VaR*-based approaches to quantify market risk they are exposed to. The horizon, for which the risk should be monitored is set to ten days[2] and pre-set confidence is .

Assuming a random variable X following a Gaussian distribution,[3] VaR over a time length Δt at confidence level α (i.e. on a probability level $p = 1 - \alpha$) can be obtained as follows:

$$VaR_X(\Delta t, \alpha) = -F_X^{-1}(1-\alpha) = -\mu_X(\Delta t) - \sigma_X(\Delta t) F_N^{-1}(1-\alpha). \qquad (1)$$

Here, $-F_X^{-1}(p)$ denotes the inverse function to the distribution function[4] of random variable X for p , which is further decomposed into the mean (the expected value) of random variable X over Δt , $\mu_X(\Delta t)$, and the product of its standard deviation, $\sigma_X(\Delta t)$, and $F_N(p)$ – p -th percentile of standard normal distribution (Gaussian distribution with zero mean and unit variance).

Unfortunately, real market returns are mostly far from being Gaussian and the decomposition $\sigma_X(\Delta t) F_N^{-1}(1-\alpha)$ is therefore not applicable if Gaussian distribution is not suitable at all and an alternative distribution is not the one with a tractable distribution function, one can apply either Monte Carlo simulation or historical simulation. While the latter is based on utilizing the series of real market returns realized in the past – and this is why it is problematic to attain a really high confidence level, the former provide us much more freedom, concerning both high confidence levels and model adaptability.

2.2 Rating-based approach

Ideally, the top management of financial institutions acts in line of equity holders' interest within bounds set by regulator and supervisor bodies. Equity holders are usually attracted by maximizing the market value of equity (i.e. stock price) which is supposed to be the best proxy to the present value of all future cash flows, due to all presently known information. Moreover, there is a need for generally acceptable and reliable measure of credible entities. The most common measure, regardless some recent events of misidentification, is rating issued by well-based agencies such as Moody's or Standard&Poor's.

Although rating assigning procedure is very complex, a crucial factor to be taken into account is the probability of default – the probability that a given entity will not be able (or willing) to meet its liabilities during a given time length, say the following year. Rating agencies regularly provide historical probabilities of defaults for each rating category. According to the mix of equity and debt holders, the entity's policy should be in accordance

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[2]As a principle, the market risk is quantified for a trading book – the portfolio of liquid and tradable instruments, for which quick rebalancing of positions is feasible.

[3]Due to its simplifying nature and overwhelming usage, the Gaussian distribution is also denoted as the *normal* distribution.

[4]To simplify the notation, $F_X^{-1}(p)$ is the inverse to the distribution function, if it exists, or at least its generalization $F_X^{-1}(p) = \inf\{x: F(x) \geq p\}$.

with the target rating category and to keep its risk of default at the implied probability of default $PD^{(i)}$.

Hence, the risk and capital sources should be managed in such a way that the true probability of default PD will neither exceed the implied one, $PD^{(i)}$, nor fall below it. While the latter would be too expensive for entity, the former would be too risky (and expensive) for the stakeholders. It follows that the entity should be able to cover all (unexpected) losses which can arise with probability $p^{(i)} = PD^{(i)}$ with available capital and thus:

$$UL_X\left(\Delta t, p^{(i)}\right) = -F_X^{-1}\left(p^{(i)}\right) + \mu_X(\Delta t). \tag{2}$$

The capital to cover unexpected losses due to the internal requirements is referred to as the *economic capital*.

3 Lévy marginals for portfolio modeling

In order to assess the risk of a portfolio, i.e. unexpected changes in its value, a joint probability distribution of all relevant drivers of random evolution should be estimated, though marginal distributions and a suitable tool to express the dependency among particular factors can be estimated separately.

Actually, such decomposition can be of great value since joint probability distribution generally presumes identical margins, at least at elementary levels. By contrast, choosing e.g. copula functions to rebuild independent marginal distributions into dependent structure gives us a great portion of freedom when estimating the marginal probability distribution.

3.1 Marginal distribution by subordinated Lévy processes

The major task of financial model building is to allow one to fit also extreme evolution of market prices. It is a matter of fact that returns at financial markets are neither symmetrically distributed nor without exceed peeks (or heavy tails) over time, which is in contradiction to Gaussian distribution. A very feasible way to fit both skewness (non-symmetry) and kurtosis (heavy tails) is to apply a subordinated Lévy model, a rather non-standard definition of Lévy models as time changed Brownian motions, which goes back to Mandelbrot and Taylor (1967) and Clark (1973).

Generally, a Lévy process is a stochastic process, which is zero at origin, its path in time is right-continuous with left limits and its main property is that it is of independent and stationary increments. Another common feature is a so called stochastic continuity. Moreover, the related probability distribution must be infinitely divisible. The crucial theorem is the Lévy-Khintchine formula:

$$\Phi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} \left(\exp(iux) - 1 - iux\mathbb{I}_{|x|<1}\right) \nu(dx). \tag{3}$$

For a given infinitely divisible distribution, we can define the triplet of Lévy characteristics,

$$\{\gamma, \sigma^2, \nu(\mathbf{dx})\}.$$

The former two define the drift of the process (deterministic part) and its diffusion. The latter is a Lévy measure. If it can be formulated as $\nu(dx) = u(x)dx$, it is a Lévy density. It is similarly to the probability density, with the exception that it need not be integrable and zero at origin. The first focus at Lévy models with jumps goes back to 1930's. The most recent and complete monographs on the theory behind and/or application of Lévy models are Kyprianou et al. (2005), Applebaum (2004), Cont and Tankov (2004), Barndorff-Nielsen et al. (2001) and Bertoin (1998).

Define a stochastic process $Z(t; \mu, \sigma)$, which is a Wiener process, as long as $\mu = 1$ and $\sigma = \sqrt{t}$, its increment within infinitesimal time length dt can be expressed as:

$$\mathrm{d}Z = \varepsilon\sqrt{dt}, \qquad \varepsilon \in \mathcal{N}[0,1], \tag{4}$$

where $\mathcal{N}[0,1]$ denotes Gaussian distribution with zero mean and unit variance. Then, a subordinated Lévy model can be defined as a Brownian motion driven by another Lévy process $\ell(t)$ with unit mean and positive variance κ . The only restriction for such a driving process is that it is non-decreasing on a given interval and has bounded variation.

Hence, we replace standard time t in

$$X(t;\mu,\sigma) = \mu dt + \sigma Z(t) \tag{5}$$

by its function $\ell(t)$:

$$X(\ell(t);\theta,\vartheta) = \theta\ell(t) + \vartheta Z(\ell(t)) = \theta\ell(t) + \vartheta\varepsilon\sqrt{\ell(t)}. \tag{6}$$

Due to its simplicity (tempred stable subordinators with known density function in the closed form), the most suitable models seem to be either the variance gamma model – the overall process is driven by gamma process from gamma distribution with parameters of shape a and scale b depending solely on variance κ , $G[a,b]$, or normal inverse Gaussian model – the subordinator is defined by inverse Gaussian model based on inverse Gaussian distribution, $IG[a,b]$. For more details on variance gamma model see e.g. Madan and Seneta (1990) (for symmetric case) and Madan and Milne (1991) and Madan et al. (1998) (for asymmetric case). Similarly, normal inverse Gaussian (NIG) model is due to Barndorff-Nielsen (1995) and (1998). Note also, that there exist several generalizations and extensions, see any of the monographs referred to above.

3.2 Dependency modeling by copula approach

A useful tool of dependency modeling are the copula functions, i.e. the projection of the dependency among particular distribution functions into $[0,1]$,

$$\mathcal{C}:[0,1]^n \to [0,1] \text{ on } \mathbb{R}^n, \qquad n \in \{2,3,\ldots\}. \tag{7}$$

Actually, any copula function can be regarded as a multidimensional distribution function with marginals in the form of standardized uniform distribution. In this paper, we restricted ourself to ordinary copula functions. Basic reference for the theory of copula functions is Nelsen (2006), while Rank (2007) and Cherubini et al. (2004) target mainly on the application issues in finance. Alternatively, Lévy processes can be coupled on the basis of Lévy measures by Lévy copula functions.

For simplicity assume two potentially dependent random variables with marginal distribution functions F_X, F_Y and joint distribution function $F_{X,Y}$. Then, following the Sklar's theorem:

$$F_{X,Y}(x,y) = \mathcal{C}(F_X(x), F_Y(y)). \tag{8}$$

If both F_X, F_Y are continuous a copula function $\mathcal{C}$ is unique. Sklar's theorem implies also an inverse relation,

$$\mathcal{C}(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)).$$

Formulation (8) above should be understood such that the joint distribution function gives us two distinct information: (i) marginal distribution of random variables, (ii) dependency function of distributions. Hence, while the former is given by $F_X(x)$ and $F_Y(y)$, a copula function specifies the dependency, nothing less, nothing more. That is, only when we put both information together, we have sufficient knowledge about the pair of random variables X, Y.

Thus, assuming that the marginal distribution functions of random variables are already known, the only further think we need to know to model the overall evolution is an appropriate copula function. With some simplification, we can distinguish copulas in the form of elliptical distributions and copulas from the Archimedean family. The main difference between these two forms is given by the ways of construction and estimation. While for the latter the primary assumption is to define the generator function, for the former the knowledge of related joint distribution function is sufficient.

3.3 Parameter estimation

There exist three main approaches to parameter estimation for copula function based dependency modeling: exact maximum likelihood method (EMLM), inference for margins (IFM), and canonical maximum likelihood (CML). While for the former all parameters are estimated within one step, which might be very time consuming, mainly for high dimensional problems or complicated marginal distributions, the latter two methods are based on estimating the parameters for the marginal distribution and parameters for the copula function separately. While assuming IFM, marginal distributions are estimated in the first step and the copula function in the second one, for CML instead of parametric margins empirical distributions are used. On more details see any of the empirically oriented literature such as Cherubini et al. (2004). In this paper we will assume IFM approach.

4 Data set

The data set we consider in our study comprises of daily effective FX rates for EUR, GBP, HUF, PLN, SKK, and USD with respect to CZK as published by the Czech National Bank, i.e. generally the market quotes at 2 p.m. The data observation started on January 1, 2000 and finished on December 31, 2008. It follows that we have at our disposal 2268 observations of log-returns for six distinct FX rates. For each FX rate basic descriptive statistics – mean, variance, standard deviation, skewness and kurtosis – of daily log-returns (per annum, if applicable) were evaluated, see Table 1.

Table 1: Descriptive statistics of daily log-returns (p.a.)

Parameter	EUR	GBP	HUF	PLN	SKK	USD
mean	−0.032	−0.079	−0.037	−0.032	0.005	−0.068
variance	0.0037	0.0088	0.0074	0.0106	0.0043	0.0135
st.deviation	0.061	0.094	0.086	0.103	0.065	0.116
skewness	−0.131	−0.236	−0.737	−0.410	0.209	0.029
kurtosis	13.010	9.357	9.951	11.511	11.192	6.829

It is apparent, that the mean returns p.a. varies substantially between -8% (GBP) and 0.5% (SKK). The only FX rate with positive drift is SKK. Similarly, also the variance of returns is various. For two FX rates we get values close to (SKK, EUR), another two are close to (GBP, HUF) and the last two go above (PLN, USD).

A positive drift of SKK implies significantly positive skewness, while the other four FX rates (EUR, GBP, HUF, PLN) are more or less negatively skewed. Next, USD returns seem to be highly symmetric. Concerning the frequency of extreme movements, all FX rates should be regarded as significantly leptokurtic, although its magnitude differs. When testing if the distribution can be regarded to be the Gaussian, several tests of Jarque-Bera type can be used. Here, the hypothesis of normality must be strongly rejected for all FX rates, mainly due to higher than normal probability of extreme movements.

Since a similar study (Tichý, 2008) was carried for the same data except the last year (2008), we can stress the main differences: standard deviations increased by approximately (in absolute values), the excess kurtosis generally increased two to three times (except HUF and PLN), mainly due to large number of extreme market movements, predominantly returns with positive sign (depreciation of CZK), which further implied decreases in the magnitude of negative skew.

In this study we focus first of all at the ability of elliptical copula functions to estimate the risk of FX rate portfolio properly. Hence, the risk/return tradeoff is not so much important as the presence of skewness and kurtosis and their mix in a notional portfolio. We therefore normalize all data to get zero mean and unit variance of log-returns either for the whole length of the data or for particular subintervals, depending on the task.

Portfolio modeling issues require some information about mutual dependencies among particular components. Although the linear correlation measure is far from perfect, which is highlighted when the underlying data are not elliptically distributed, it is still of high information value. We therefore report the correlation matrix bellow:

$$\mathbf{R} = \begin{pmatrix} 1. & 0.58 & 0.39 & 0.25 & 0.71 & 0.47 \\ 0.58 & 1. & 0.26 & 0.35 & 0.45 & 0.65 \\ 0.39 & 0.26 & 1. & 0.48 & 0.45 & 0.14 \\ 0.25 & 0.35 & 0.46 & 1. & 0.34 & 0.31 \\ 0.71 & 0.37 & 0.45 & 0.34 & 1. & 0.31 \\ 0.47 & 0.65 & 0.14 & 0.31 & 0.31 & 1 \end{pmatrix}.$$

We can observe that the correlation coefficient is always positive, for most cases between 0.31 and 0.47 , with three exceptions outside these bounds (to both sides). As one might guess, the highest correlation is for (EUR, SKK) and (GBP, USD) – i.e. the pair of economies with tight linkage (lowest, respectively) to the Czech one.

5 Results

Let us assume a unit of financial institution that is responsible for trading with foreign currencies and hedging of open positions. Assume next that the residual portfolio (i.e. open positions) consists of six distinct currencies as in Table 1, for simplicity each with equal weight $w = 1/6$. Domestic currency is CZK, the overall amount is 1 000 000 CZK.

Table 2: Risk measures of equally weighted portfolio estimated over 2000-2008 (*df* = 7)

Parameter	2000-2008	GBM-G	VG-G	NIG-G	GBM-St	VG-St	NIG-St
mean	-0,00017	-0,00017	-0,00016	-0,00017	-0,00016	-0,00018	-0,00017
variance	0,00001	0,00001	0,00001	0,00001	0,00001	0,00001	0,00001
st.deviation	0,00355	0,00355	0,00349	0,00351	0,00355	0,00353	0,00354
skewness	-0,467	-0,001	-0,221	-0,213	0,004	-0,275	-0,301
kurtosis	6,582	2,993	4,777	4,608	3,443	6,627	6,578

In Table 2 we provide the empirical features of the data series (basic descriptive statistics) and the ones obtained by selected models, applying Monte Carlo simulation technique: GBM for (geometric) Brownian motion, VG for variance gamma model, NIG for Normal Inverse Gaussian model, while G states for Gaussian copula and St for Student copula function. Since only the assumption of Student copula function provides us with a quite well estimation of kurtosis, we will work further with this model only.

An important part of risk management is to examine the sensitivity of the risk – and simultaneously, the amount of capital required to cover it – to input factors. From the point of view of FX rate sensitive portfolio, the most significance inputs are standard deviation, skewness, and kurtosis of particular risk drivers (FX rates) and the dependency among them. We change all parameters by k 10%, $k = –0.5, –0.4, ..., 0,4, 0.5$. Thus, for overall portfolio we obtain standard deviations from 0.17% to 0.53%, levels of skewness from –0.12 to –0.45, and levels of kurtosis from 4 to 8. Concerning the dependency, we modify the degrees of freedom, ie the measure of linearity of dependency. The results are depicted in Figure 1 (we assume only GBM-St, VG-St and NIG-St models – VaR for p = 0.01% (disk) and p = 0.0003 (diamond), cVaR for 0.01% (square) and 0.0003 (triangle).

As it might be expected, the GBM models, even if coupled by Student copula function, is insensitive to the modification of skewness and kurtosis, almost insensitive to *df*. We will therefore make comments mainly on VG-St and NIG-St models.

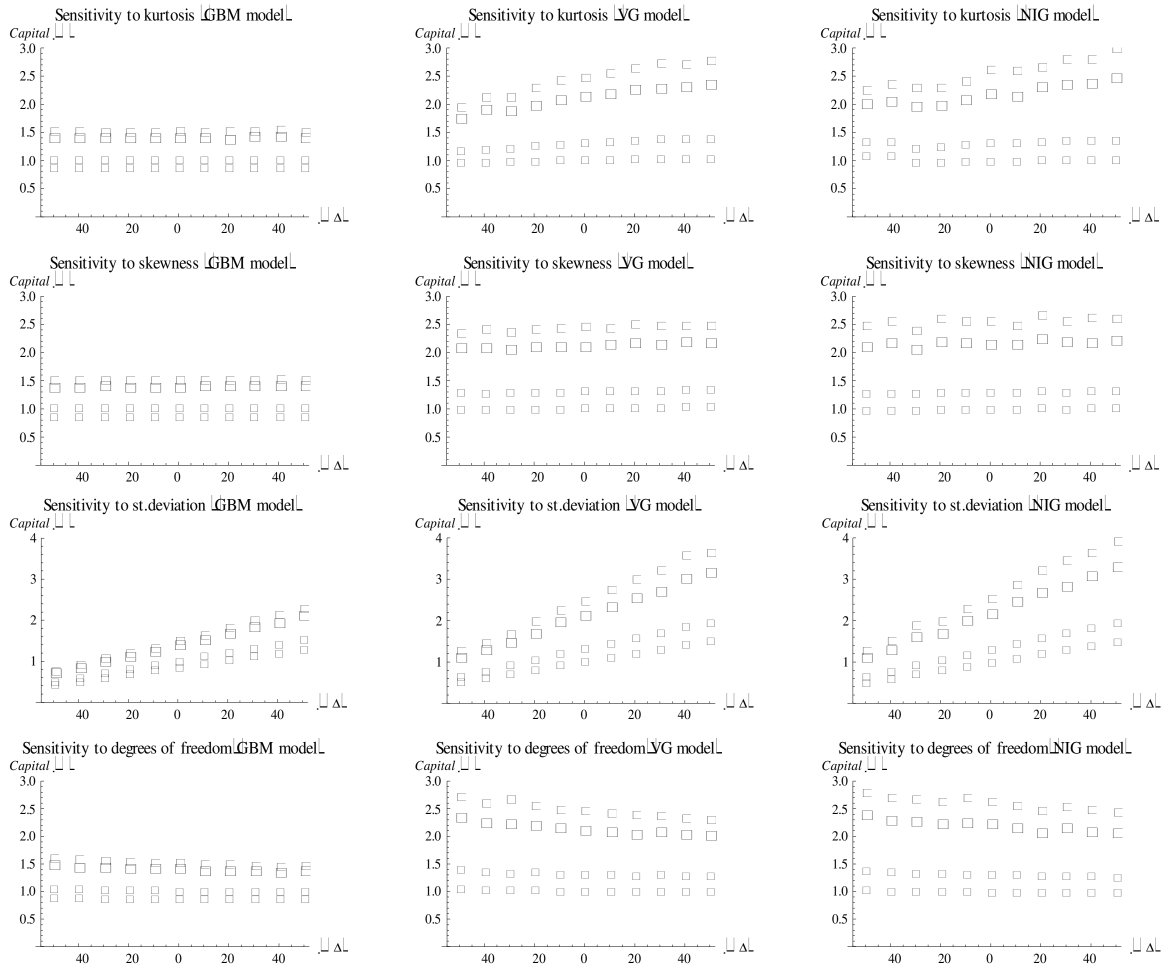

Figure 1: Sensitivity of capital requirements to selected input parameters

As one might guess, the effect of kurtosis modification is evident mainly for p = 0.0003 (results for NIG model are not very smooth). Hence, even if the probability of extreme events increases (the tails are heavier), as we could observe during the last year, the capital requirement due to Basel II approach is rather insignificant. However, for internal management purposes and subsequent capital allocation among particular units the level of kurtosis is important (up to 20% of additional capital). The sensitivity of chosen risk measures to skewness modification is rather low, mainly due to the data (negative skewness) and portfolio composition. By contrast, the modification of standard deviation implies very sharp changes in the amount of required capital.

6 Conclusion

An efficient management of financial risks can increase the performance of any given entity. Moreover, the risk measurement and subsequent capital allocation can be used to increase competitiveness among particular units of a given institutions. In this paper we have shown a powerful tool to estimate the (market) risk of a portfolio – the standard copula subordinated Lévy model. We have also calculated the sensitivity of selected risk measures to the modification of input factors. This analysis is important in order to asses a vulnerability of financial institutions (or their units) to the change of external factors.

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