Inflation rate prediction – a statistical approach

Předpověď míry inflace - statistický přístup

František Vávra[1], Tomáš Ťoupal[2], Eva Wagnerová[3], Patrice Marek[4], Zdeněk Hanzal[5]

Abstract

This paper deals with the prediction of inflation rate expressed by several types of indices. The statistical approach is applied, i.e. we assume the inflation to have the same probabilistic behaviour in the predicted period as in the covered past. Such approach can be applied for one type of prediction. Moreover, it is applicable for future testing whether the observation is influenced by a different effect than those influential in the period of collecting data for the statistical inference. The structure of the paper is the following. The main part contains basic relations and results whereas more detailed derivations are stated in the appendix.

Key words

Inflation rate, price index, probabilistic model, parameter estimate, inflation rate prediction

JEL Classification: C13, C53

1 Measuring Inflation

Inflation is a multidimensional and complex phenomenon. One of its projections is the Consumer Price Index (CPI) that we will employ to present the results of this paper. Other possible measures of inflation are indices of construction works and buildings price, indices of producer price in industry or agriculture, indices of market services price, etc.

These indices can also be differentiated according to time. Primarily, there are:

- Index "previous period = 100 %"

$$i(t+1,t)=\frac{c_{t+1}}{c_t}, \qquad \textbf{(1)}$$

where

c_t is the price of the given "market basket" at the time t, usual time unit is one month.

It is therefore a classic chain index.

- Index "same period of the previous year = 100 %"

$$i(t+\tau,t)=\frac{c_{t+\tau}}{c_t}, \qquad \textbf{(2)}$$

where

τ is equal to twelve when time is measured in months.

It is therefore a sort of basic index.

[1] doc. Ing. František Vávra, CSc., katedra matematiky, Fakulta aplikovaných věd Západočeské univerzity v Plzni, vavra@kma.zcu.cz

[2] Ing. Tomáš Ťoupal, katedra matematiky, Fakulta aplikovaných věd Západočeské univerzity v Plzni, ttoupal@kma.zcu.cz

[3] Ing. Eva Wagnerová, katedra matematiky, Fakulta aplikovaných věd Západočeské univerzity v Plzni,ewa@kma.zcu.cz

[4] Ing. Patrice Marek, katedra matematiky, Fakulta aplikovaných věd Západočeské univerzity v Plzni,patrke@kma.zcu.cz

[5] Ing. Zdeněk Hanzal, katedra matematiky, Fakulta aplikovaných věd Západočeské univerzity v Plzni,zhanzal@kma.zcu.cz

- Index "base period = 100 %"

$$i(t,t_0)=\frac{c_t}{c_{t_o}},\qquad \textbf{(3)}$$

where

c_{t_o} is the price of the given (goods, price, commodity) "market basket" in the comparative period t_0. This price may not be determined to a specific point in time, it can be related to a whole period, e.g. "the average of a given year = 100 %".

Clearly, this is a classic basic index.

To avoid the influence of immediate extremes, different types of averaging are used, e.g. an average index (backward moving average)

$$I(t,\upsilon)=\frac{1}{\upsilon}\sum_{j=0}^{\upsilon-1}\frac{c_{t-j}}{c_{t-j-\tau}},\ \text{usually}\ I(t,\tau)=\frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}.\qquad \textbf{(4)}$$

This is a very brief selection subjectively focused on what we will use further. The following trivial relations (ČSÚ, 2011) will be also useful.

$$i(t+\tau,t)=\frac{c_{t+\tau}}{c_t}=\prod_{j=1}^{\tau}\frac{c_{t+j}}{c_{t+j-1}}=\prod_{j=1}^{\tau}i(t+j,t+j-1)\qquad \textbf{(5)}$$

$$i(t+\tau,t)=\frac{c_{t+\tau}}{c_t}=\frac{\frac{c_{t+\tau}}{c_{t_o}}}{\frac{c_t}{c_{t_o}}}=\frac{i(t+\tau,t_0)}{i(t,t_0)}\qquad \textbf{(6)}$$

For values of $i(t+1,t)$ that do not differ too much from one ($0{,}90<i(t+1,t)<1{,}10$), the following estimate will be of use (for details see the appendix):

$$I(t,\tau)=\frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}\approx\sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}}=\sqrt[\tau]{\prod_{j=0}^{\tau-1}i(t-j,t-j-\tau)}=\sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{i(t-j,t_0)}{i(t-j-\tau,t_0)}}\ \textbf{(7)}$$

2 Models and statistical inference

The relation (5) is a starting point for the model of above mentioned inflation rates. It is well known that the random variable $\eta_t=\lg i(t,t-1)$ can be described by a normal probability distribution. Moreover, the observations very often form a non-correlated time series. Then, if these empirical observations hold, also $\lg i(t+\tau,t)=\sum_{j=1}^{\tau}\lg i(t+j,t+j-1)$ is normally distributed.

2.1 Statistical inference of η_t

To identify the parameters, we used monthly data from (ČNB, 2011) spanning the period January 2010 – April 2011. However, using the classic approach of parameter estimation (setting the mean and variance equal to the sample estimates) we could not accept the hypothesis of consistency with a normal probability model. After several attempts to solve this issue we decided to estimate the parameters minimizing the criteria:

$$\sum_{i=1}^{n}\left|F_e(y_i)-\Phi\left(\frac{y_i-\mu}{\sigma}\right)\right|\underset{\mu,\sigma}{\rightarrow}\min,$$

where

$F_e(y_i)$ is the value of the empirical distribution function in the i-th observation y_i of the random variable η_t and

$\Phi(z)$ is the cumulative distribution function of the standard normal distribution *N(0,1)*, i.e. $\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{x^2}{2}}dx$.

This new method has then given an acceptable result which is presented by the following figures 1, 2, 3 and table 1.

Figure 1: Comparison of edf and model cdfs for both types of parameter estimates.

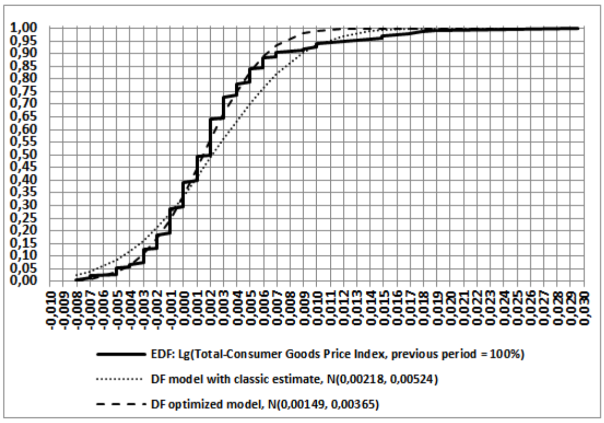
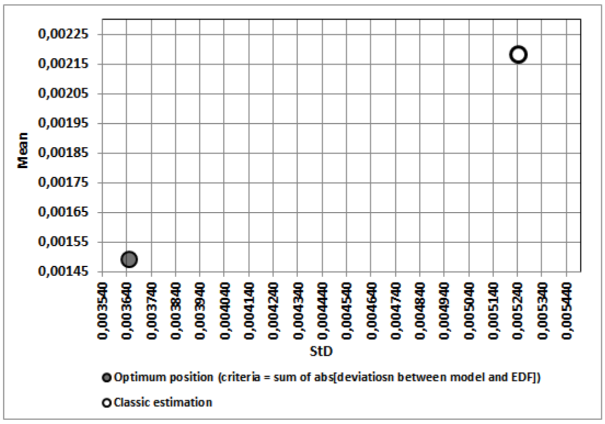


Figure 2: Comparison of position of both types of parameter estimates.

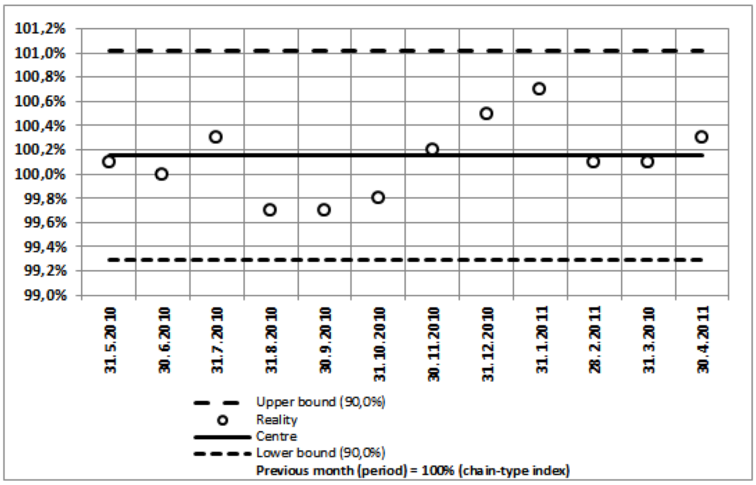


Parameter estimate	Classic	Optimized
Mean: lg(TOTAL-Consumer Goods Price Index, previous period = 100 %)	0,00218	0,00149
StD: lg(TOTAL-Consumer Goods Price Index, previous period = 100 %)	0,00524	0,00365

Table 1: Parameter values for both types of estimates.

Further, we tested the empirical presumption of non-correlation from the sample correlation coefficients between individual months in a year. The assumption of zero correlation was accepted on the significance level of 2 %.

Figure 3: Example of 90% limits, median and particular observations of the random variable , i.e. .



2.2 Models of Other Rates

Under the stated and verified assumptions, the formulas (5) – (7) transform the description of random behaviour of particular inflation rates to models derived from a normal distribution .

2.2.1 Index "previous period = 100 %"

For this index, the relation holds. Also, for such random variable, it holds with given significance levels , where

$\alpha_1 = \Phi\left(\frac{d_{0,\alpha_1}-\mu}{\sigma}\right) \Rightarrow d_{0,\alpha_1} = \mu + \sigma\Phi^{-1}(\alpha_1),$

$1-\alpha_2 = \Phi\left(\frac{h_{0,\alpha_2}-\mu}{\sigma}\right) \Rightarrow h_{0,\alpha_2} = \mu + \sigma\Phi^{-1}(1-\alpha_2),$

where $\Phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution.
However,

$$\alpha = P\left(d_{0,\alpha_1} < lgi(t+\Delta, t+\Delta-1) < h_{0,\alpha_2}\right) = P\left(e^{d_{0,\alpha_1}} < i(t+\Delta, t+\Delta-1) < e^{h_{0,\alpha_2}}\right),$$

therefore

$e^{d_{0,\alpha_1}}$ is the α_1 lower bound for the random variable $i(t+\Delta, t+\Delta-1)$, **(8)**

$e^{h_{0,\alpha_2}}$ is the α_2 upper bound for the random variable $i(t+\Delta, t+\Delta-1)$. **(9)**

From the above it follows that the random variable $i(t+\Delta, t+\Delta-1)$ has a log-normal distribution. This somewhat complicates the representation of point predictions mean. If we choose $\alpha_1 = \alpha_2 = 0.5$, we obtain $e^{d_{0,0.5}} = e^{h_{0,0.5}}$ which is the median that can be used as the point estimate.

2.2.2 Index "same period of the previous year = 100 %"

At first, we will deal with the case $\Delta \le \tau$. It holds $i(t+\Delta, t+\Delta-\tau) = \frac{c_{t+\Delta}}{c_{t+\Delta-\tau}} = \frac{\frac{c_t}{c_{t-\tau}}}{\frac{c_{t-\tau+\Delta}}{c_{t-\tau}}}\frac{c_{t+\Delta}}{c_t}$. However, under the stated assumption the term $\frac{\frac{c_t}{c_{t-\tau}}}{\frac{c_{t-\tau+\Delta}}{c_{t-\tau}}}$ is known and only the term $\frac{c_{t+\Delta}}{c_t} = \prod_{j=1}^{\Delta}\frac{c_{t+j}}{c_{t+j-1}}$ is random. The random variable $\lg i(t+\Delta, t) = \sum_{j=1}^{\Delta}\lg \eta_{t+j}$ is normally distributed with the mean $\Delta\mu$ and standard deviation $\sqrt{\Delta}\sigma$, i.e. $N\left(\Delta\mu, \sqrt{\Delta}\sigma\right)$. Then, it is just a numerical technique to determine the confidence intervals for $\frac{c_{t+\Delta}}{c_t}$ analogically as in the subchapter 2.2.1.

Now, the case $\Delta > \tau; i.e. \Delta = \tau + k; k = 1,2 \ldots$ It holds $i(t+\Delta, t+\Delta-\tau) = \frac{c_{t+\tau+k}}{c_{t+k}} = \prod_{j=1}^{\tau}\frac{c_{t+j}}{c_{t+j-1}}$. The random variable $\lg i(t+\Delta, t) = \sum_{j=1}^{\Delta}\lg\eta_{t+j}$ is then normally distributed with the mean $\tau\mu$ and standard deviation $\sqrt{\tau}\sigma$, i.e. $N\left(\tau\mu, \sqrt{\tau}\sigma\right)$. For details see (P1) – (P4) in the appendix.

2.2.3 Index [average index (backward moving average)]

This index has the form $I(t,\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}$. The probability model of this rate is more complicated than in the sub-chapter 2.2.2, case $\Delta \le \tau$. The source of complications is the fact that the backward averaging causes the elements $\eta_t = \lg i(t, t-1)$ to appear several times in the random variable. The derivation and explicit forms are in the appendix, relations (P6) – (P11).

3 Experiments and results

The following graphs show the results of our experiments and computations. For all of the three mentioned types of indices, we demonstrate the known values, one year prediction, the median curve and 90% limits. Figure 4 shows the index "same period of the previous year = 100 %" and figure 5 the "average of the year 2005 = 100 %" index. On the fig. 6 there is the "backward moving average" index. For comparison, there is also a prediction of the Czech National Bank (CNB)[6]. Its wider limits are given by the fact that this prediction rises from an

[6] Source: *http://www.cnb.cz/cs/menova_politika/prognoza/index.html*.

econometric model. It presumes the index to develop accordingly with other macroeconomic quantities. On the other hand, the statistical approach presumes the externalities to keep the same influence on the inflation.

Figure 4: "Same period of the previous year = 100 %" index.

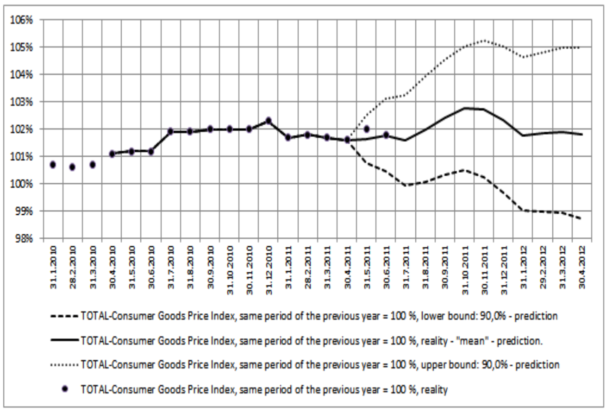


Figure 5: "Average of the year 2005 = 100%" index.

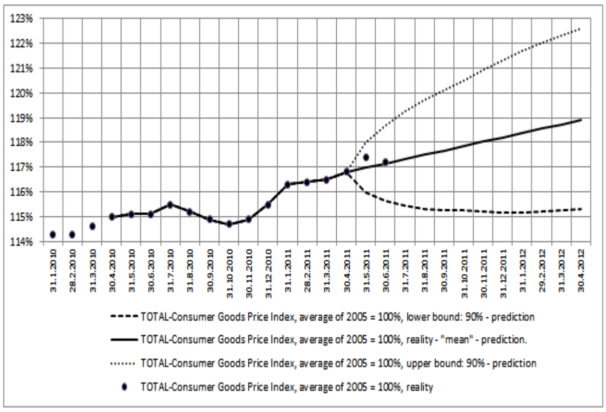
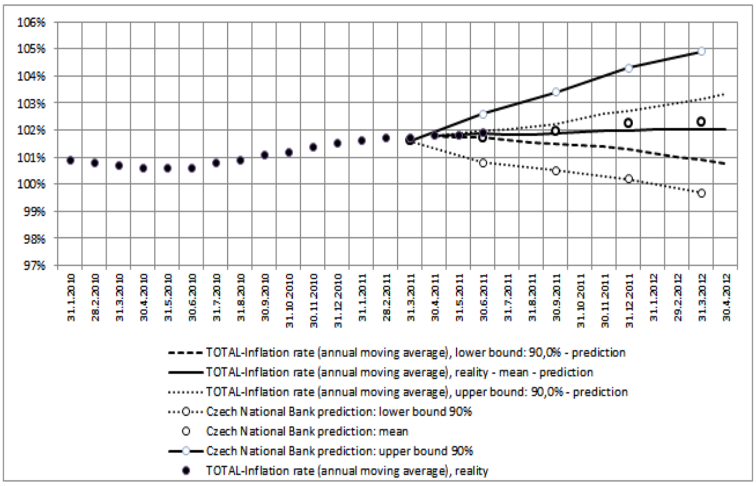


Figure 6: Average index (backward moving average).



4 Summary

In this paper we derived methods of prediction for different types of inflation rates. We also stated conditions for using the methodology. We presented a new concept of estimating the probability models parameters and the use of the geometric mean to approximate the arithmetic one. The statistical approach can be applied both for forecasting and future testing of whether the effects of other macroeconomic variables on the inflation remain unchanged.

References

[1] KUFNER, A., 1975. *Nerovnosti a odhady.* Praha: Mladá fronta.

[2] RÉNYI, A., 1972. *Teorie pravděpodobnosti.* Praha: Academia.

[3] Česká národní banka (ČNB), 2011. ARAD – Systém časových řad. [online] Available at: <http://www.cnb.cz/cnb/STAT.ARADY_PKG.STROM_DRILL?p_strid=CDAA&p_lang=CS> [Accessed 26th May 2011]

[4] Český statistický úřad (ČSÚ), 2011. Míra inflace. [online] Available at:

<http://www.czso.cz/csu/redakce.nsf/i/mira_inflace> [Accessed 26th May 2011]

Summary

V předkládaném příspěvku jsou odvozeny metody předpovědi jednotlivých typů měr inflace. Jsou uvedeny i předpoklady použití dané metodiky. Za nové lze považovat pojetí odhadů parametrů pravděpodobnostního modelu a aproximaci aritmetického průměru geometrickým. Statistické postupy lze využít jednak pro vlastní předpověď ale také i pro budoucí testy toho, zda se nemění působení ostatních makroekonomických veličin.

Appendix – derivation of the applied methods

Since the basic index is expressible by the chain indices: $i(t+\Delta,t)=\frac{c_{t+\Delta}}{c_t}=\prod_{j=1}^{\Delta}\frac{c_{t+j}}{c_{t+j-1}}$, $\Delta=1,2,\ldots$, its logarithm has the form: $l_{t,\Delta}=lg\big(i(t+\Delta,t)\big)=\sum_{j=1}^{\Delta}lg\left(\frac{c_{t+j}}{c_{t+j-1}}\right)$. Let us now assume the random variable $\eta_t=\lg\left(\frac{c_t}{c_{t-1}}\right)$ to be stationary in a broader sense, i.e. $\mathrm{E}\{\eta_t\}=\mu$, $\sigma^2\{\eta_t\}=\sigma^2$ and $\mathrm{corr}\{\eta_t,\eta_s\}=0\Leftrightarrow t\neq s$ and to have a normal distribution. Then, the variable $l_{t,\Delta}$ is also normally distributed and $l_{t,\Delta}\approx N(\mu\tau,\sigma^2\Delta)$. It is therefore a Gaussian random walk in a broader sense. Then, the tolerance interval in which the value of $l_{t,\Delta}$ lies with a probability $1-(\alpha_1+\alpha_2)$ can be written as

$$1-(\alpha_1+\alpha_2)=P\left(d_{\alpha_1}(\Delta)<l_{t,\Delta}<h_{\alpha_2}(\Delta)\right), \qquad \textbf{(P1)}$$

where the numbers $d_{\alpha_1}(\Delta)<h_{\alpha_2}(\Delta)$ are given by the equalities $\alpha_1=P\left(l_{t,\Delta}\leq d_{\alpha_1}(\Delta)\right)$ and $\alpha_2=P\left(l_{t,\Delta}\geq h_{\alpha_2}(\Delta)\right)$. Further, under the stated conditions for the random variable η_t, these equalities can be written as:

$$\alpha_1=P\left(l_{t,\Delta}\leq d_{\alpha_1}(\Delta)\right)=\Phi\left(\frac{d_{\alpha_1}-\mu\Delta}{\sigma\sqrt{\Delta}}\right)\Rightarrow d_{\alpha_1}(\Delta)=\mu\Delta+\sigma\sqrt{\Delta}\Phi^{-1}(\alpha_1);\Delta=1,2,\ldots, \qquad \textbf{(P2)}$$

$$\alpha_2=P\left(l_{t,\Delta}\geq h_{\alpha_2}(\Delta)\right)=1-\Phi\left(\frac{h_{\alpha_2}(\Delta)-\mu\Delta}{\sigma\sqrt{\Delta}}\right)\Rightarrow h_{\alpha_2}(\Delta)=\mu\Delta+\sigma\sqrt{\Delta}\Phi^{-1}(1-\alpha_2);\Delta=1,2,\ldots, \qquad \textbf{(P3)}$$

where $\Phi^{-1}(\alpha)$ is the $\alpha-$quantile of the standard normal distribution, i.e. $\Phi^{-1}(\alpha)$ is a solution of the equation $\alpha=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{x^2}{2}}dx$ with respect to z. Given the fact that the exponential is a strictly increasing function we obtain:

$$1-(\alpha_1+\alpha_2)=P\left(d_{\alpha_1}(\Delta)<l_{t,\tau}<h_{\alpha_2}(\Delta)\right)=P\left(e^{d_{\alpha_1}(\Delta)}<\frac{c_{t+\Delta}}{c_t}<e^{h_{\alpha_2}(\Delta)}\right), \qquad \textbf{(P4)}$$

therefore the values $e^{d_{\alpha_1}(\Delta)}<e^{h_{\alpha_2}(\Delta)}$ are the $1-(\alpha_1+\alpha_2)$ tolerance interval for the basic index $i(t+\Delta,t)=\frac{c_{t+\Delta}}{c_t}$. Thence for the chain index it holds:

$$1-(\alpha_1+\alpha_2)=P\left(d_{\alpha_1}(1)<l_{t,1}<h_{\alpha_2}(1)\right)=P\left(e^{d_{\alpha_1}(1)}<\frac{c_{t+1}}{c_t}<e^{h_{\alpha_2}(1)}\right).$$

The equations (P2) and (P3) can be used for a "point" prediction by the choice $\alpha_1=\alpha_2=0.5$. Then $d_{0,5}(\Delta)=h_{0,5}(\Delta)=\mu\Delta$, because $\Phi^{-1}(0,5)=0$. Therefore:

$$P\left(\frac{c_{t+\Delta}}{c_t}\leq e^{\mu\Delta}\right)=P\left(\frac{c_{t+\Delta}}{c_t}\geq e^{\mu\Delta}\right)=0.5. \qquad \textbf{(P5)}$$

Similarly, for the chain index $P\left(\frac{c_{t+1}}{c_t}\leq e^{\mu}\right)=P\left(\frac{c_{t+1}}{c_t}\geq e^{\mu}\right)=0.5$.

Sometimes, the backward moving average of the basic indices is used for demonstration, clearly:

$$I(t,\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}$$

In the case of $\frac{c_{t-j}}{c_{t-j-\tau}} \approx 1$, the value of $I(t,\tau)$ can be approximated by a geometric mean: $I(t,\tau) \cong \sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}}$, more precisely $I(t,\tau) \geq \sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}}$ which is a known inequality between the arithmetic and geometric mean, see [1]. For $\tau = 12$ and $0.9 \leq \frac{c_{t-j}}{c_{t-j-\tau}} \leq 1.1$, the approximation error of $\sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{c_{t-j}}{c_{t-j-\tau}}}$ with respect to $I(t,\tau)$ will be less than 0.0022 with the certainty of 95 %. We will notice later the advantage of this seemingly complicated approximation.

The prediction form of a backward moving average of basic indices in given periods for the time Δ from now is $I(t+\Delta,\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{c_{t+\Delta-j}}{c_{t+\Delta-j-\tau}}$. Using the geometric mean approximation we obtain $I(t+\Delta,\tau) \cong \sqrt[\tau]{\prod_{j=0}^{\tau-1}\frac{c_{t+\Delta-j}}{c_{t+\Delta-j-\tau}}}$. However we can write $\frac{c_{t+\Delta-j}}{c_{t+\Delta-j-\tau}} = \prod_{k=1}^{\tau}\frac{c_{t+\Delta-j-\tau+k}}{c_{t+\Delta-j-\tau+k-1}}$, therefore $I(t+\Delta,\tau) \cong \sqrt[\tau]{\prod_{j=0}^{\tau-1}\prod_{k=1}^{\tau}\frac{c_{t+\Delta-j-\tau+k}}{c_{t+\Delta-j-\tau+k-1}}}$. From this expression we obtain $\lg\big(I(t+\Delta,\tau)\big) \cong \frac{1}{\tau}\sum_{j=0}^{\tau-1}\sum_{k=1}^{\tau}\lg\left(\frac{c_{t+\Delta-j-\tau+k}}{c_{t+\Delta-j-\tau+k-1}}\right) = \frac{1}{\tau}\sum_{j=0}^{\tau-1}\sum_{k=1}^{\tau}\eta_{t+\Delta-j-\tau+k} = \frac{1}{\tau}\sum_{l=2-\tau}^{\tau}(\tau-|l-1|)\eta_{t+\Delta-\tau+l}$.

In the time t, some elements of the sum in the definition

$$lg\big(I(t+\Delta,\tau)\big) = \frac{1}{\tau}\sum_{l=2-\tau}^{\tau}(\tau-|l-1|)\eta_{t+\Delta-\tau+l} \qquad \textbf{(P6)}$$

are known and some are predicted. We predict those $\eta_{l+t+\Delta-\tau}$, for which $l+t+\Delta-\tau > t$, thus $l > \tau-\Delta$. Therefore

$$\lg\big(I(t+\Delta,\tau)\big) = \frac{1}{\tau}\sum_{l=2-\tau}^{\tau-\Delta}(\tau-|l-1|)\eta_{t+\Delta-\tau+l} + \frac{1}{\tau}\sum_{l=\tau-\Delta+1}^{\tau}(\tau-|l-1|)\eta_{t+\Delta-\tau+l}$$
$$= \lg known\,(\tau,t+\Delta) + \lg foreseen\,(\tau,t+\Delta),$$

where

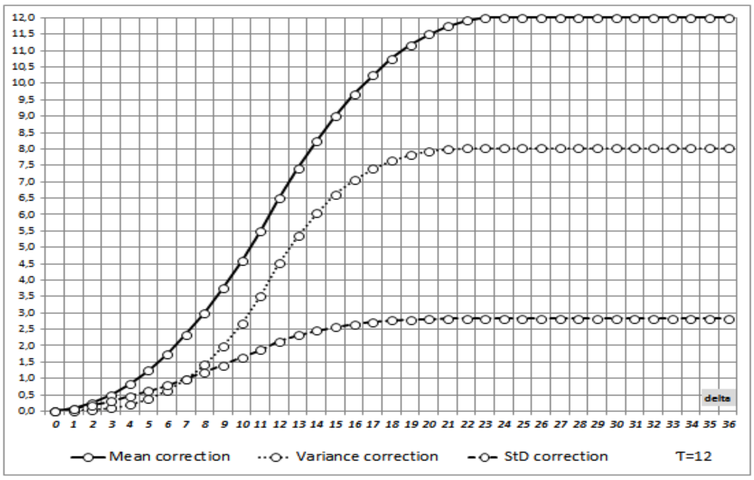
$$\lg known\,(\tau,t+\Delta) = \frac{1}{\tau}\sum_{l=2-\tau}^{\tau-\Delta}(\tau-|l-1|)\eta_{t+\Delta-\tau+l} \text{ and} \qquad \textbf{(P7)}$$
$$\lg foreseen\,(\tau,t+\Delta) = \frac{1}{\tau}\sum_{l=\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)\eta_{t+\Delta-\tau+l}\,.$$

We use the convention $\sum_{l=d}^{h}c_l = 0$, if $h < d$.
From here
$E\{\lg foreseen\,(\tau,t+\Delta)\} = \frac{\mu}{\tau}\sum_{\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|) = \mu k_{\mu}(\tau,\Delta)$,
$\sigma^2\{\lg foreseen\,(\tau,t+\Delta)\} = \frac{\sigma^2}{\tau^2}\sum_{\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)^2 = \sigma^2 k_{\sigma^2}(\tau,\Delta)$,
$\sigma\{\lg foreseen\,(\tau,t+\Delta)\} = \frac{\sigma}{\tau}\sqrt{\sum_{\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)^2} = \sigma k_{\sigma}(\tau,\Delta)$ and the variable $foreseen(\tau,t+\Delta)$ has a normal distribution with the above parameters, where

$$k_{\mu}(\tau,\Delta) = \frac{\sum_{l=\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)}{\tau}, \qquad \textbf{(P8)}$$
$$k_{\sigma^2}(\tau,\Delta) = \frac{\sum_{l=\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)^2}{\tau^2} \text{ and}$$
$$k_{\sigma}(\tau,\Delta) = \sqrt{k_{\sigma^2}(\tau,\Delta)} = \frac{\sqrt{\sum_{l=\max(2-\tau,\tau-\Delta+1)}^{\tau}(\tau-|l-1|)^2}}{\tau}.$$

In some cases, the correction coefficients (P8) can be expressed analytically by adding up the sums (e.g. , …). However, their forms are complicated (especially for) and the definition forms with sums are more practical. The course of these coefficients is on the figure P1.

Figure P1: The course of the correction coefficients , , .



Similarly to the derivation of (P1) – (P4) we have

, **(P9)**

where the values of are given by equalities and

.

Again, given the conditions on the random variable we can write these equalities in the forms:

(P10)

(P11)

The inequality (P9) "inside the probability" can be modified without the change of the set of possible solutions, as follows:

=

where

$known(\tau, t+\Delta) = exp\left(\frac{1}{\tau}\sum_{l=2-\tau}^{\tau-\Delta}(\tau - |l-1|)\eta_{t+\Delta-\tau+l}\right) = exp\left(\frac{1}{\tau}\sum_{j=0}^{2(\tau-1)-\Delta}(\tau - |\tau - \Delta - j - 1|)\eta_{t-j}\right) = exp\left(\frac{1}{\tau}\sum_{j=0}^{2(\tau-1)-\Delta}(\tau - |\tau - \Delta - j - 1|)\lg\left(\frac{c_{t-j}}{c_{t-j-1}}\right)\right).$