MODELING OF EXPECTED RETURNS WITHIN GBM AND VG MODELS

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ABSTRAKT

Modelování očekávaných výnosů je nedílnou součástí finančního modelování. Obecně existují dva přístupy k výpočtu (modelování) výnosů aktiv. Je možné postupovat buď přímo dle stochastické diferenciální rovnice nebo dle modelu, který vyjadřuje cenu aktiva a výnosy zpětně dopočíst. Nevýhodou druhého přístupu je, že při nevhodné aplikaci může vést k výrazné chybě vzhledem k odlišné střední hodnotě. V článku jsou oba přístupy popsány a vysvětleny na případě normálně rozložených výnosů (geometrický Brownův pohyb) a moderního modelu zohledujícího i šikmost a špičatost (variance gama model). Teoretická argumentace je podpořena simulačním příkladem.

ABSTRACT

Modeling of expected returns is an inherent step of financial modeling. In general, there exist two approaches to calculate the asset return. We can proceed either directly according to a stochastic differential equation or due to a model which describes asset price evolution. However, the latter case should be applied carefully, since it can lead to computation error due to the mean correcting term. In this paper we describe both approaches in detail and derive an error which can arise. We suppose two distinct models: standard model of normally distributed returns (geometric Brownian motion) and an example of modern models family, which respect also skewness and kurtosis (Variance gamma model). We support the theoretical argumentation by several simulation results.

Introduction

Financial modeling play very important role in financial decision-making of all subjects. It can be useful when future levels of financial quantities are required for decision making procedure.

An inherent step of financial modeling is to calculate returns of financial asset prices. The returns can be modeled either directly via application of suitable stochastic differential equation or indirectly via modeling of the asset price evolution first. However, the latter case can lead to significant errors, since it aims first of all on the future price, but not the return. The result will be incorrect due to the mean correcting term. By contrast, this approach is useful when some higher order moments of the returns distribution are calculated (e.g. the variance does not depend on the constant).

In this paper we formulate two basic stochastic differential equations and related solution for modeling of the asset price evolution. We also derive expected increments for each case. Furthermore, we explain why the exponential term in the latter case should

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be different to the stochastic differential equation. We also provide the instructions how to calculate the error of inadequately used model for returns modeling.

All of that is produced for both, the geometric Brownian motion and Variance gamma model. These models are defined in Section 2, the arguments, we spoke above, are derived in Section 3. In Section 4 we run Monte Carlo simulation in order to verify formulas derived in Section 3. It also helps us to clarify our theoretical results. Furthermore, we provide simple example of digital option static replication error (on the basis of simulated returns).

1. Stochastic processes

Stochastic processes are standard tools used in financial modeling. The future price of any financial asset which is regarded to be riskless can be determined in advance. Any other asset, future price of which can be described by suitable probability distribution, is regarded to be risky. In this case, we can model the future price of such asset by means of stochastic process on the basis of probability distribution.

1.1 Normally distributed returns

In finance, standard assumption is to suppose that the prices of financial quantities are of *lognormal distribution*. Such distribution can be defined for random variable x by the following *distribution function* F_{LN}:

$$F_{LN}(x) = \int_0^x \frac{\exp[-\frac{1}{2}\left(\frac{\ln z - m}{s}\right)^2]}{z\sqrt{2\pi}s}\mathrm{d}z, \tag{1}$$

or by *density function* f_{LN}:

$$f_{LN}(x) = \frac{\exp[-\frac{1}{2}\left(\frac{\ln x - m}{s}\right)^2]}{x\sqrt{2\pi}s}. \tag{2}$$

Here, m denotes the mean value of $\ln(x)$ and s is its standard deviation. The big advantage of models based on lognormal distribution is that the prices can be only positive – there is no upper limit, the price can rise infinitely, but the zero (or negative) value is not admissible.

Due to the well known property of the lognormal distribution[2] – it is followed by the exponential of the normally distributed random variables – we also know that the (continuous-time) returns of lognormally distributed asset price $\mathcal{S}$,

$$\mu = \ln\frac{\mathcal{S}_{t+dt}}{\mathcal{S}_t}, \tag{3}$$

follows the normal distribution. Note, that dt denotes the infinitesimal time interval.

The normal distribution is defined by its distribution function $F_{\mathcal{N}}$ (μ is the average return (drift) and σ is its volatility):

$$F_{\mathcal{N}}(x) = \int_{-\infty}^x f_{\mathcal{N}}(z)\mathrm{d}z = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^x \exp\left[-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right]\mathrm{d}z \tag{4}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-x}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right]\mathrm{d}z, \tag{5}$$

[2] If a variable is lognormally distributed, its natural logarithm will be distributed normally. If the natural logarithm of variable $\mathcal{S}$ is normally distributed, the sum (or difference) of natural logarithms should be also normally distributed.

or *density function* $f_{\mathcal{N}}$:

$$f_{\mathcal{N}}(x) = \frac{\exp[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2]}{\sqrt{2\pi}\sigma}. \tag{6}$$

Analyzing the density function of the normal distribution and lognormal distribution we can discover the following relations. If the random variable is distributed according to the law of the lognormal distribution, its natural logarithm is distributed normally. Moreover, the distribution function of lognormal distribution can be defined on the basis of the normal distribution function:

$$F_{LN}(x) = F_{\mathcal{N}}\left(\frac{\ln x - m}{s}\right). \tag{7}$$

Thus, the asset price $\mathcal{S} \in LN[m, s]$ with $m = \mathbb{E}[\ln \mathcal{S}]$ and $s^2 = \mathtt{var}[\ln \mathcal{S}]$. Simultaneously, $\ln \mathcal{S} \in \mathcal{N}[\mu, \sigma]$ with $\mu = \mathbb{E}[\ln \mathcal{S}]$ and $\sigma^2 = \mathtt{var}[\ln \mathcal{S}]$. Now, it is obvious that the lognormal distribution is defined in the terms of the mean and the variance of related normal distribution and we can therefore replace s and m by μ and σ, respectively. Finally, we should not miss the fact that the lognormal distribution is not defined by its mean $\mathbb{E}[\mathcal{S}] = e^{\mu+\frac{\sigma^2}{2}}$ and variance $\mathtt{var}[\mathcal{S}] = \sigma^2\mathbb{E}[\mathcal{S}]^2$ (we explain these relations in the next section in more details).

The model which is based on the normal distribution of asset returns or, equivalently, lognormal distribution of asset prices, is usually referred to as the *geometric Brownian motion* (GBM):[3]

$$\mathcal{S}_{t+dt}^{(\mathbb{P})} = \mathcal{S}_t \exp\left[(\mu - \omega)\, dt + \sigma\sqrt{dt}\epsilon\right] = \mathcal{S}_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\sqrt{dt}\epsilon\right]. \tag{8}$$

It is the solution of the following stochastic differential equation (SDE):

$$d\mathcal{S}^{(\mathbb{P})} = \mu\mathcal{S}_t dt + \sigma\mathcal{S}_t d\mathcal{Z}_t. \tag{9}$$

Here, $d\mathcal{Z}_t$ is a Wiener process $\mathcal{Z}_t = \varepsilon\sqrt{t}$, $\varepsilon \in \mathcal{N}(0; 1)$. The term $\omega = \frac{\sigma^2}{2}$ must be deduced in (8) in order to correct the mean of the model, and $(\mathbb{P})$ indicates, that the model is defined under the real world measure (not risk-neutral one). In this way, the expected return will really be μ. The justification will be provided later. Note, that it was rigorously derived by K. Itô [5].

1.2 Non-normally distributed returns

Modern models of financial asset price evolution try to respect the well known fact, see e.g. Fama [3], that the asset returns are not distributed normally – exhibit excess kurtosis and either positive or (more often) negative skewness. Several such models defined as subordinated Brownian motions belongs to the Lévy models family.

Denote $\mathcal{Z}(t; \sigma, \mu)$ as time t dependent Wiener process with parameters $\mu = 1$ and $\sigma = \sqrt{t}$, i.e. $\mathcal{Z}_t = \varepsilon\sqrt{t}$, $\varepsilon \in \mathcal{N}(0; 1)$. Then, we can define Brownian motion $\mathcal{X}(t; \theta, \vartheta)$ with increment θ and volatility ϑ driven by Lévy process $\ell(t)$ is simple – we just need to replace t by $\ell(t)$. Hence:

$$\mathcal{X}_t = \theta\ell(t) + \vartheta\mathcal{Z}(\ell_t), \tag{10}$$

[3] Since the model was (re)introduced into the finance area in Black and Scholes [1], the terms Black and Scholes world and Black and Scholes setting also refer to this model.

which can be also formulated as follows:

$$\mathcal{X}_t = \theta\ell(t) + \vartheta\varepsilon\sqrt{\ell(t)}. \tag{11}$$

We can interpret this formula in such a way that increment $d\mathcal{X}$ within infinitesimal time interval dt is normally distributed with (random) mean value $\theta\ell(dt)$ and variance $\mathtt{var} = \vartheta^2\ell(dt)$. The mean value of the subordinating process $\ell(t)$ should be dt and its variance will govern the *fat tails* of the distribution.

Here, we will state as an example the *Variance gamma model* (VG model) introduced subsequently by Madan and Seneta [8] (symmetric case), Madan and Milne [7] and Madan *et al.* [6] (asymmetric case). For more details on Lévy models family or definition/application of other processes see e.g. Cont and Tankov or Tichý [9].

The VG model is driven by *gamma process*. The probability density function of gamma distribution $\mathcal{G}[\mu;\nu]$ with $\mu = 1$ is defined as follows:

$$f_{\mathcal{G}}(g, t; \mu = 1, \nu) = \frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}. \tag{12}$$

Since $\mathcal{VG}(g(t;\nu);\theta,\vartheta)$ can be defined as

$$\mathcal{VG}_t = \theta g_t + \vartheta\mathcal{Z}(g_t) = \theta g_t + \vartheta\sqrt{g_t}\varepsilon, \tag{13}$$

we can get the density function of the VG model as follows:

$$f_{\mathcal{VG}}(\mathcal{X}, g(t;\nu);\theta,\vartheta) = \int_0^\infty \frac{1}{\vartheta\sqrt{2\pi g}}\exp\left(-\frac{(\mathcal{X}-\theta g)^2}{2\vartheta^2 g}\right)\frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}\,\mathrm{d}g. \tag{14}$$

On this basis, we can formulate the distribution function:

$$F_{\mathcal{VG}}(\mathcal{X}, g(t;\nu);\theta,\vartheta) = \int_{-\infty}^{\mathcal{X}}\int_0^\infty \frac{1}{\vartheta\sqrt{2\pi g}}\exp\left(-\frac{(z-\theta g)^2}{2\vartheta^2 g}\right)\frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}\,\mathrm{d}g\mathrm{d}z. \tag{15}$$

Finally, the asset price model can be formulated as follows:

$$\mathcal{S}_{t+dt}^{(\mathbb{P})} = \mathcal{S}_t\exp\left(\mu dt + \mathcal{VG}_{dt}^{(\mathbb{P})} - \omega dt\right) = \mathcal{S}_t\exp\left(\mu dt + \theta g_{dt} + \vartheta\sqrt{g_{dt}}\varepsilon - \omega dt\right), \tag{16}$$

where $\omega = -\frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\vartheta^2\nu\right)$.

2. Modeling of the expected return

The key step in financial modeling is to be able to estimate the future evolution of asset prices or theirs returns in both, the risk-neutral and real market conditions.

2.1 Geometric Brownian motion

Recall GBM (8):

$$\mathcal{S}_{t+dt} = \mathcal{S}_t\exp\left[(\mu-\omega)\,dt + \sigma\sqrt{dt}\epsilon\right] = \mathcal{S}_t\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}\epsilon\right]$$

and related SDE (9):

$$d\mathcal{S} = \mu \mathcal{S}_t dt + \sigma \mathcal{S}_t d\mathcal{Z}.$$

Since both equations are valid also for longer time intervals, we can replace dt by $T - t = \tau > 0$.

In order to get the price at time T, it is sufficient to produce random number from standard normal distribution, $\epsilon \in \mathcal{N}[0,1]$, and put it into (8) to get $\mathcal{S}_T^{(i)}$. Here, the upper index (i) indicates the $i-$th scenario. If we repeat the procedure sufficiently many times, the mean of the produced data set should correspond to results obtained analytically due to (8) and (9).

Thus,

$$\mathbb{E}[\mathcal{S}_T] = \frac{1}{N}\sum_{i=1}^{N} \mathcal{S}_T^{(i)}.$$

According to (8), we have:

$$\mathbb{E}[\mathcal{S}_T] = \mathbb{E}\left[\mathcal{S}_t e^{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\epsilon}\right]$$

This can be decomposed as follows:

$$\mathbb{E}[\mathcal{S}_T] = \mathbb{E}\left[\mathcal{S}_t e^{\mu\tau} e^{-\frac{\sigma^2}{2}\tau} e^{\sigma\sqrt{\tau}\epsilon}\right].$$

Here, the initial asset price is non-random, as well as the first two exponents. Hence, we can write:

$$\mathbb{E}[\mathcal{S}_T] = \mathcal{S}_t e^{\mu\tau} e^{-\frac{\sigma^2}{2}\tau} \mathbb{E}\left[e^{\sigma\sqrt{\tau}\epsilon}\right].$$

Although the expected value of ϵ is zero, the same is not true for $\exp(\epsilon)$. Due to the properties of the exponential function and the standard normal distribution, we get:

$$\mathbb{E}\left[e^{\sigma\sqrt{\tau}\epsilon}\right] = e^{-\frac{\sigma^2}{2}\tau}.$$

Obviously, we can conclude, that the expected value of $\mathcal{S}_T$ is determined by nothing more than μ and τ:

$$\mathbb{E}[\mathcal{S}_T] = \mathcal{S}_t e^{\mu\tau}.$$

It also implies, that:

$$\ln \frac{\mathbb{E}[\mathcal{S}_T]}{\mathcal{S}_t} = \mu\tau.$$

Here, we should stress commonly produced mistake (and usually overlooked). If instead of taken the mean of the terminal price we calculate the returns first,

$$\mathbb{E}[x] = \frac{1}{N}\sum_{i=1}^{N} \ln \frac{\mathcal{S}_T^{(i)}}{\mathcal{S}_t},$$

we incorrectly arrive at:

$$\mathbb{E}[x] = \left(\mu - \frac{\sigma^2}{2}\right)\tau.$$

It results from the fact, that

$$\mathbb{E}[x] = \mathbb{E}\left[\ln \frac{\mathcal{S}_t e^{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\epsilon}}{\mathcal{S}_t}\right] = \mathbb{E}\left[\ln\left(e^{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\epsilon}\right)\right] = \mathbb{E}\left[\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\epsilon\right].$$

Clearly, since the expected value of ϵ is zero, we get again:

$$\mathbb{E}[x] = \left(\mu - \frac{\sigma^2}{2}\right)\tau.$$

Of course, if we are interested in modeling of the return for some purposes, it is sufficient to proceed according to SDE (9). In this case, the mean will be, as we want, $\mu\tau$. Within this context, we should also stress that if we are interested in the variance, we should either proceed according to (9) and calculate only returns, or alternatively due to (8). In the latter case, we can get either variance of returns (by taking natural logarithms of particular prices) or variance of prices. In both cases, we get the same variance of returns since only the random part, identical in both formulation, is significant.

2.2 Variance gamma model

Within the VG model, the procedure is similar. In order to get the future price of an asset, we must according to (16) produce two independent random numbers ε and g, each from distinct distribution, standard normal and gamma.

$$\mathcal{S}_T^{(\mathbb{P})} = \mathcal{S}_t \exp\left(\mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \omega\tau\right).$$

Repeating the procedure several times, we can calculate the mean due to

$$\mathbb{E}[\mathcal{S}_T] = \frac{1}{N}\sum_{i=1}^{N} \mathcal{S}_T^{(i)}.$$

Obviously, it should be equal to:

$$\mathbb{E}[\mathcal{S}_T] = \mathbb{E}\left[\mathcal{S}_t e^{\mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \omega\tau}\right].$$

Since $\mathbb{E}[\exp(\theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon)] = \exp(\omega\tau)$, we can simplify this formula as follows:

$$\mathbb{E}[\mathcal{S}_T] = \mathbb{E}\left[\mathcal{S}_t e^{\mu\tau} e^{\theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon} e^{-\omega\tau}\right] = \mathbb{E}\left[\mathcal{S}_t e^{\mu\tau}\right] = \mathcal{S}_t e^{\mu\tau},$$

which corresponds to

$$\ln \frac{\mathbb{E}[\mathcal{S}_T]}{\mathcal{S}_t} = \mu\tau.$$

Here, we can also calculate the returns before taking the expectation. In such case we proceed as follows:

$$\begin{aligned}
\mathbb{E}[x] &= \mathbb{E}\left[\ln \frac{\mathcal{S}_t e^{\mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \omega\tau}}{\mathcal{S}_t}\right] \\
&= \mathbb{E}\left[\ln\left(e^{\mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \omega\tau}\right)\right] \\
&= \mathbb{E}\left[\mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \omega\tau\right] \\
&= \mu\tau - \omega\tau + \mathbb{E}\left[\theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon\right] \\
&= \mu\tau - \omega\tau + \theta\tau.
\end{aligned}$$

This is clearly incorrect since $\omega \neq \theta$. (We utilize the fact, that $\mathbb{E}[\epsilon] = 0$ and $\mathbb{E}[g_\tau] = \tau$.)

It also implies, that if we want to model the returns directly, we could simply use neither the exponential part of (16) reduced by ω or (13). In the latter case we must add the drift μ and deduce the mean of $\mathcal{VG}$, which is $\theta\tau$, while in the former, it is sufficient to omit the mean correcting term ω and deduce $\theta\tau$:

$$\tilde{x} = \mu\tau + \theta g_\tau + \vartheta\sqrt{g_\tau}\varepsilon - \theta\tau.$$

3. (Un)expected return and option replication

In this section, we first verify by means of Monte Carlo simulation the argumentation concerning the returns which was derived in the preceding section. Next, we examine the sensitivity of static digital option replication by tight spread on the real world returns, within both GBM and VG model.

3.1 Simulation of returns

Suppose the nondividend stock-like asset with initial price $\mathcal{S}_0 = 100$, the time horizon $\tau = 0.1$, the riskless rate $r = 0.05$ p.a. Suppose also, that the average return of such asset $\mu = 0.10$ with volatility $\sigma = 0.25$, both on annul basis. Furthermore (VG model), we will suppose the skewness -0.80 and kurtosis 4.14.

First, we will produce N random scenarios (subsequently for $N = 1\ 000,\ 10\ 000$ and $100\ 000$) according to GBM (8). On the basis of the formulas above, we should get either

$$\mathbb{E}[\mathcal{S}_T] = \mathcal{S}_0 \exp(\mu\tau) = 100 \times \exp(0.1 \times 0.1) = 101.005$$

or

$$\mathbb{E}[x] = \frac{1}{N}\sum_{i=1}^{N} \ln \frac{\mathcal{S}_T^{(i)}}{\mathcal{S}_t} = \left(\mu - \frac{\sigma^2}{2}\right)\tau = \left(0.1 - \frac{0.25^2}{2}\right)0.1 = 0.006875.$$

Similarly, the variance should be $\sigma^2\tau = 0.00625$.

The results are included in Table 1. We can see that with several thousands of independent scenarios the simulation works efficiently. Note, that in order to get columns 3 and 5, we can use directly (9).

Table 1: Geometric Brownian motion

N of scenarios N	*price* $\mathbb{E}[\mathcal{S}_T]$	*return* $\ln\frac{\mathbb{E}[\mathcal{S}_T]}{\mathcal{S}_0}$	*incorrect return* $\mathbb{E}[\ln\frac{\mathcal{S}_T}{\mathcal{S}_0}]$	*variance*
1 000	101.011	0.010056	0.006921	0.006272
10 000	101.005	0.009999	0.006875	0.006250
100 000	101.005	0.010000	0.006875	0.006250

Now, we will do the same (N random scenarios for $N = 1\ 000,\ 10\ 000,\ 100\ 000$, and $1\ 000\ 000$) for the VG model.[4] On the basis of the formulas above, we should get either

$$\mathbb{E}[\mathcal{S}_T] = \mathcal{S}_0 \exp(\mu\tau) = 100 \times \exp(0.1 \times 0.1) = 101.005$$

[4] In order to fit the skewness and kurtosis we have to set $\theta = -0.34$, $\vartheta = 0.19$ and $\nu = 0.228$, which gives $\omega = -0.3107$.

or

$$\mathbb{E}[x] = \frac{1}{N}\sum_{i=1}^{N} \ln \frac{\mathcal{S}_T^{(i)}}{\mathcal{S}_t} = (\mu - \omega + \theta)\,\tau = (0.1 + 0.3107 - 0.34)\,0.1 = 0.007068.$$

The variance should stay the same: 0.00625.

The results are included in Table 2. We can see that in order to get the proper expected price, we should run several hundred of thousands of independent scenarios. (Poor convergence can be observed also for other characteristics.)

In order to estimate the expected return, we can use two variants, as indicated above. We can either get the price due to (16), take the mean and calculate the continuously defined return (V1), or, alternatively, we can produce VG-random number due to (13), take its mean, subtract its theoretical value and add the drift μ (V2). With large N, both results should be the same. As it is obvious from the table, that is not true in general. The reason is that in both variants, the simulation error is differently specified, either as $\mathbb{E}[\exp(\mathcal{VG})]$ (V1) or $\mathbb{E}[\mathcal{VG}]$.

By contrast, since the variance does not depends on the constant terms, it is the same independently on the approach we choose. The same is true for skewness and kurtosis which are not reproduced here (the convergence is slightly worse comparing to the variance).

Table 2: Variance gamma model

N of scenarios N	*price* $\mathbb{E}[\mathcal{S}_T]$	*return V1* $\ln \frac{\mathbb{E}[\mathcal{S}_T]}{\mathcal{S}_0}$	*return V2* $\mathbb{E}[\mathcal{VG}] + (\mu - \theta)\tau$	*incorrect return* $\mathbb{E}[\ln \frac{\mathcal{S}_T}{\mathcal{S}_0}]$	*variance*
1 000	100.831	0.00828	0.00812	0.00519	0.006617
10 000	100.969	0.00964	0.00956	0.00663	0.006405
100 000	100.979	0.00975	0.00973	0.00680	0.006283
1 000 000	101.008	0.01002	0.01002	0.00709	0.006246

3.2 Digital option replication

A digital option is a special type of financial derivative which pays off either everything ($\Psi = Q$) or nothing ($\Psi = 0$) – the payoff function is discontinuous. This feature slightly complicates pricing and hedging issues. In general, we can distinguish two basic approaches to pricing and hedging of financial derivatives – dynamic replication and static decomposition. The former is based on keeping of everchanging portfolio of the risky and riskless asset. This approach is strongly model-dependent. By contrast, static approach is based on decomposition of the digital option into several liquid assets, potentially other derivatives. This approach should be model-free. In this part of the paper, we examine if this statement holds also for VG model.

Suppose digital cash-or-nothing call option $\mathcal{V}_{call}^{dig/cash}(\tau; \mathcal{S}, \mathcal{K}; Q)$. Here, τ is time to maturity ($\tau = 0.1$), $\mathcal{S}$ and $\mathcal{K}$ are the underlying asset price and exercise price, respectively, both set at the level of 100 initially, and for simplicity, the payoff amount $Q = 1$. This option can be statically decomposed into $1/\alpha$ tight spreads of vanilla call options $\mathcal{H}_{(\mathcal{K}-\alpha;\mathcal{K})}^{vanillacall}$:

$$\mathcal{H}_{(\mathcal{K}-\alpha;\mathcal{K})}^{vanillacall} = \mathcal{H}_{call}^{vanilla}(\tau; \mathcal{S}, \mathcal{K} - \alpha) - \mathcal{H}_{call}^{vanilla}(\tau; \mathcal{S}, \mathcal{K})$$

For more details see e.g. [10].

Parameter α will control the (theoretical) error bounds. However, it would be chosen with respect to market conditions. Here, we suppose that plain vanilla options are liquid at the market for $\alpha = 0.5$. It implies the replication error at maturity for the region $\mathcal{S}_T \in [99.5, 100]$ as follows: $\mathcal{E} \in [0, 1]$, see Figure 1.

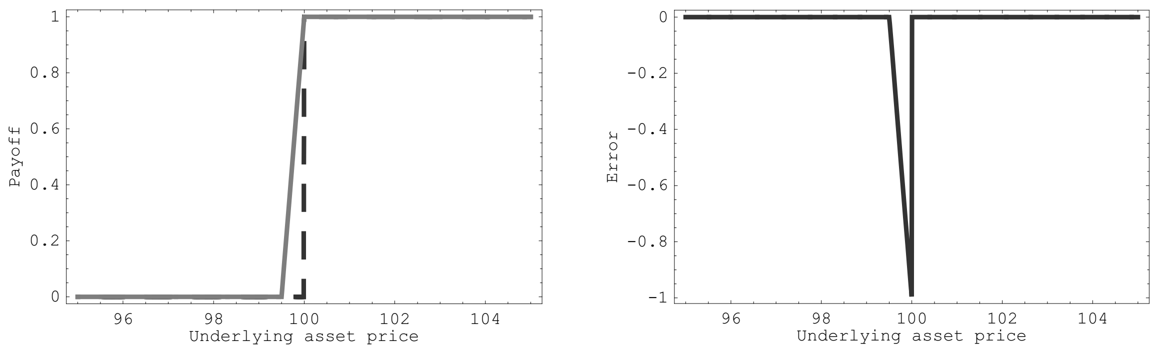

Figure 1: Payoff of the digital call (dashed black line) and replicating portfolio (grey solid line) on the left; replication error on the right

Now, we will analyze the effect of four distinct drifts ($\mu = 0.00$, $\mu = 0.05$, $\mu = 0.10$, and $\mu = 0.15$) on the digital option replication error within both GBM and VG models. This time, however, we produce $N = 100\ 000$ random scenarios. More particularly, we study the probability of the error and basic characteristics of its distribution (shape, mean and standard deviation and minimum and maximum values). These results are provided in Table 3 (GBM) and Table 4 (VG).

Table 3: Static replication of digital option – error parameters for GBM

Parameter	$\mu = 0.00$	$\mu = 0.05$	$\mu = 0.10$	$\mu = 0.15$
$Pr[\mathcal{E} < 0]$	0.25	0.25	0.25	0.25
$\min[\mathcal{E}]$	-1	-1	-1	-1
$\max[\mathcal{E}]$	0	0	0	0
mean$[\mathcal{E}]$	-0.500	-0.500	-0.500	-0.500
st.dev$[\mathcal{E}]$	0.289	0.289	0.289	0.289

Table 4: Static replication of digital option – error parameters for VG

Parameter	$\mu = 0.00$	$\mu = 0.05$	$\mu = 0.10$	$\mu = 0.15$
$Pr[\mathcal{E} < 0]$	0.24	0.21	0.21	0.19
$\min[\mathcal{E}]$	-1	-1	-1	-1
$\max[\mathcal{E}]$	0	0	0	0
mean$[\mathcal{E}]$	-0.510	-0.517	-0.508	-0.516
st.dev$[\mathcal{E}]$	0.291	0.291	0.291	0.291

Concerning the GBM, it is clear that true drift of the underlying asset price does not play (almost) any role. The probability, that the error will arise is approximately 0.25 and it is distributed between zero and minus one (as expected due to the theoretical boundaries).

The probability distribution shape (due to the lack of place we do not provide the chart here) indicates, that the error is uniformly distributed between its boundaries (0 and -1), the mean as well as median is 0.5. These characteristics are given mainly by the fact, that the option is ATM and the drift has almost no impact on the probability distribution of $\mathcal{S} \in [99.5, 100]$.

By contrast, it is probably not the case of the VG model – through the skewness, the drift moves the probability distribution so that the probability of error slightly changes. Particular errors are distributed between the boundaries (0 and -1), however, not so equally as in the case of GBM.

Conclusions

In this paper we have tried to clarify the modeling of asset prices and their returns via GBM and VG models and relevant stochastic differential equations. We have presented several formulas, showing us the way to calculate the asset return which will be in accordance with the expected evolution. We have also derived errors of inadequately applied approaches.

Finally, we have verified the theoretical results by Monte Carlo simulation. Simple example of digital option static replication was also included in the paper.

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