DEPENDENCY STRUCTURE MODELS OF FINANCIAL ASSET RETURNS

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ABSTRAKT

Nedílnou součástí modelování portfolia, risk managementu, oceňování opcí a obdobných problémů finančního inženýrství je poskytnout pravdivý obrázek o závislosti koše rizikových aktiv. Dle standardních předpokladů Blacka a Scholese, tedy při normálním rozložení výnosů finančních aktiv, lze aplikovat Choleskyho dekompozici buď kovarianční nebo korelační matice náhodných prvků. Avšak při respektování současného stavu finančních trhů je nutné brát v vahu i vyšší momenty pravděpodobnostního rozdělení než jen střední hodnotu a rozptyl, konkrétně se jedná o šikmost a špičatost, což podstatnou měrou komplikuje i modelování závislostí. V článku je poskytnut přehled základních přístupů k modelování závislostí včetně vyšších momentů šikmosti a špičatosti. Vybraný přístup je aplikovat s cílem odhadnout pravděpodobnostní rozložení výnosů portfolia.

ABSTRACT

A very important part of portfolio modeling, risk management, option pricing, and several other issues of financial engineering is to give a true picture about the dependency of risky factor baskets. Under standard assumption of Black and Scholes model, i.e. under normally distributed returns, Cholesky decomposition of either covariance or correlation matrix of random terms can be applied. However, present day market conditions are far from this assumption, since non-zero skewness and excess kurtosis are typical features there. In this paper we review basic approaches to model the dependency of asset returns including the higher moments of skewness and kurtosis. Selected approach is applied in order to estimate the probability distribution function of portfolio returns.

Introduction

An inherent part of modern finance theory is the area of financial modeling. One of the most important issues is to model the future evolution of financial asset prices in order to calculate entire risk or to price a derivative asset. When a portfolio of financial assets is modeled, i.e. particular basket of assets, the dependency among all risky components should be taken into account.

Standard approach of portfolio dependency modeling is based on the application of the Cholesky decomposition of normally distributed random terms. Although the assumption of normally distributed returns could do quite well several dozens years ago, at present time such assumption is much far from reality. Some studies documenting non-zero skewness and higher than normal kurtosis appeared early in 60's, see e.g. Fama [7]. However, with globalization effects and due to tight links among national markets, such anomalies like

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financial crises and sudden defaults of various companies and also several governments arise more often. These events subsequently result into even higher skewness and kurtosis comparing with the levels documented in 60's and 70's.

It is not surprising that new models of financial price evolution are regularly suggested. Among the most important ones which allow us to match also higher moments of the underlying distribution, i.e. skewness and kurtosis, we can classify *the variance gamma model* (VG model) introduced subsequently by Madan and Seneta [12] (symmetric case), Madan and Milne [11] and Madan *et al.* [10] (asymmetric case), *normal inverse Gaussian model* (NIG) due to Barndorff-Nielsen (see [2] and [3]) and extensions of VG model like the CGMY model due to Carr, Geman, Madan and Yor (see [4] and also [5]).

All such models belong to a pure jump Lévy models family and consists of (at least) two independent stochastic processes. This feature evidently complicate a portfolio modeling procedure. In this paper we aim at two things. The first task is to provide a brief survey of basic approaches to dependency modeling when VG model is considered. The second task is to examine one selected approach considering the case of FX-sensitive portfolio evolution.

We proceed as follows. In the following section, basic definitions of geometric Brownian motion and variance gamma model are provided. Next, in Section 3 we state all possible combinations of modeling the dependency between two variance gamma processes. Finally, in Section 4 we choose one particular approach in order to model the evolution of three foreign exchange rates, EUR/CZK, GBP/CZK, and USD/CZK all together.

1. Stochastic processes of Lévy type

Formally, a stochastic process $\{\mathcal{X}(t), t \in [0,T]\}$ is a Lévy process on $[0,T]$, if (for $\tau \geq 0$):

1. it starts at zero: $\mathcal{X}(0) = 0$,
2. its increments are independent: $\mathcal{X}(t+\tau) - \mathcal{X}(t)$ does not depend on $\mathcal{X}(s)$, $s \leq t$,
3. its increments are stationary distributed: $\mathcal{X}(t+\tau) - \mathcal{X}(t) = \mathcal{X}(\tau)$, in other words, it depends only on τ,
4. it is stochastically continuous: $\lim_{\tau \to 0} Pr[\mathcal{X}(t+\tau) - \mathcal{X}(t) > \epsilon] = 0$ for $\epsilon > 0$.

Last but not least, the distribution is infinitely divisible. Note also, that for many Lévy models infinite intensity of possibly very small jumps is typical feature (see also property 4).

A very special type of Lévy models is the *geometric Brownian motion* (GBM), since its increments are always continuous. Although the model is standard and popular tool of financial practitioners, it has several drawbacks, coming out from the fact that it is based on the normal distribution (returns are symmetrically distributed).

The GBM,

$$\mathcal{S}_{t+dt}^{(\mathbb{P})} = \mathcal{S}_t \exp\left[(\mu - \omega)\, dt + \sigma\sqrt{dt}\epsilon\right] = \mathcal{S}_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\sqrt{dt}\epsilon\right], \tag{1}$$

is a solution to the following stochastic differential equation (SDE):

$$d\mathcal{S}^{(\mathbb{P})} = \mu \mathcal{S}_t dt + \sigma \mathcal{S}_t d\mathcal{Z}_t. \tag{2}$$

Here, μ is the long term average return, σ is its volatility, both per annum, and $d\mathcal{Z}_t$ is a Wiener process $\mathcal{Z}_t = \varepsilon\sqrt{t}$, $\varepsilon \in \mathcal{N}(0;1)$. The term $\omega = \frac{\sigma^2}{2}$ must be deduced in (1) in order to correct the mean of the model,[2] and $(\mathbb{P})$ indicates, that the model is defined under the real world measure (not risk-neutral one).

A model, which respect the up-to-date features of financial markets is *a variance gamma model* (VG model). Its original definition was suggested as a (geometric) Brownian motion in stochastically defined time. Thus, we can formulate VG model $\mathcal{VG}(g(t;\nu);\theta,\vartheta)$ easily replacing standard time scale t in (2) by a random *gamma time* g_t as follows:

$$\mathcal{VG}_t = \theta g_t + \vartheta \mathcal{Z}(g_t) = \theta g_t + \vartheta\sqrt{g_t}\varepsilon. \tag{3}$$

Obviously, since we require the prices to be only positive, the asset price model should be an exponential one:

$$\mathcal{S}^{(\mathbb{P})}_{t+dt} = \mathcal{S}_t \exp\left(\mu dt + \mathcal{VG}^{(\mathbb{P})}_{dt} - \omega dt\right) = \mathcal{S}_t \exp\left(\mu dt + \theta g_{dt} + \vartheta\sqrt{g_{dt}}\varepsilon - \omega dt\right), \tag{4}$$

where $\omega = -\frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\vartheta^2\nu\right)$. Here, as usual, μ is the long term average return and ϑ, θ, and ν are the parameters controlling VG process on the basis of a gamma-time with unit mean (on standard time t basis). An economic interpretation is that the process evolves like GBM, but in dependency on unequally distributed information. Since we model the flow of information by gamma process (i.e. stochastic time), we can say that prices changes in jumps, whenever new information arrives.

To complete the theory, we also provide first four moments of VG distribution in Table 1.

Table 1: First four moments of VG distribution

Parameter	$\mathcal{VG}(g(t;\nu);\theta,\vartheta)$
Mean	θ
Variance	$\vartheta^2 + \nu\theta^2$
Skewness	$\frac{\theta\nu(3\vartheta^2+2\nu\theta^2)}{(\vartheta^2+\nu\theta^2)^{\frac{3}{2}}}$
Kurtosis	$3\left(1 + 2\nu - \frac{\nu\vartheta^4}{(\vartheta^2+\nu\theta^2)^2}\right)$

2. Dependency modeling

In this section we provide a survey of available approaches to dependency modeling of VG distributed random variables, including an economic justification. Since the transformation of (3) into (4) is obvious, we only need to be able to match the dependencies of random terms in (3).[3]

[2] Note, that it was rigorously derived by K. Itô, see e.g. [14] for derivation.

[3] When calculating the terminal price according to (4) we add one constant and deduce another. This operation has therefore no impact on variance, correlation, skewness and kurtosis. For more on modeling of multivariate Lévy processes see e.g. [6], [9], [16] and references therein.

The following combinations can happen (g is a random number from gamma distribution $\mathcal{G}[t/\nu, \nu]$ with mean t and variance ν and ϵ is a random number from standard normal distribution $\mathcal{N}[0, 1]$):

Group A Identical g's and

- identical ϵ's,
- correlated ϵ's,
- independent ϵ's.

Group B Independent g's and

- identical ϵ's,
- correlated ϵ's,
- independent ϵ's.

Group C Correlated g's and

- identical ϵ's,
- correlated ϵ's,
- independent ϵ's.

Group D Dependency of overall VG processes, i.e. processes are mutually

- identical, $\mathcal{VG}_i \equiv \mathcal{VG}_j$,
- correlated,
- independent.

Within the latter group, we can model the dependency by a copula approach.

As a first step, it seems to be useful to derive general covariance (correlation) formula of two possibly dependent VG processes:

$$\mathcal{VG}_i = \theta_i g_i + \vartheta_i \sqrt{g_i} \varepsilon_i,$$

where $i = 1, 2$. Due to the principles of covariance calculation and utilizing the fact, that in general $\texttt{cov}[g, \epsilon] = 0$, we can write:

$$\texttt{cov}[\mathcal{VG}_1, \mathcal{VG}_2] = \theta_1 \theta_2 \texttt{cov}[g_1, g_2] + \vartheta_1 \vartheta_2 \mathbb{E}[\sqrt{g_1 g_2}] \mathbb{E}[\epsilon_1 \epsilon_2]. \tag{5}$$

Similarly, for correlation we get:

$$\begin{aligned} \texttt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] &= \frac{\texttt{cov}[\mathcal{VG}_1, \mathcal{VG}_2]}{\texttt{var}[\mathcal{VG}_1]\texttt{var}[\mathcal{VG}_2]} \\ &= \frac{\theta_1 \theta_2 \texttt{cov}[g_1, g_2] + \vartheta_1 \vartheta_2 \mathbb{E}[\sqrt{g_1 g_2}] \mathbb{E}[\epsilon_1 \epsilon_2]}{\sqrt{\vartheta_1^2 + \nu_1 \theta_1^2} \sqrt{\vartheta_2^2 + \nu_2 \theta_2^2}}. \end{aligned} \tag{6}$$

Group A

Suppose that we have only one gamma process. In this case, correlation formula (6) changes into:

$$\texttt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = \frac{\theta_1\theta_2\nu + \vartheta_1\vartheta_2\mathbb{E}[\epsilon_1\epsilon_2]}{\sqrt{\vartheta_1^2 + \nu_1\theta_1^2}\sqrt{\vartheta_2^2 + \nu_2\theta_2^2}}. \tag{7}$$

Thus, $\texttt{cov}[g_1, g_2]$ is equal to the variance of the identical gamma process ν, $\mathbb{E}[\sqrt{g_1g_2}] = 1$ and we work with θ_1, ϑ_1, θ_2, ϑ_2, and ν.

If the two Wiener processes are independent, i.e. $\mathbb{E}[\epsilon_1\epsilon_2] = 0$, the correlation coefficient can be calculated as follows (Model 1):

$$\texttt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = \frac{\theta_1\theta_2\nu}{\sqrt{\vartheta_1^2 + \nu\theta_1^2}\sqrt{\vartheta_2^2 + \nu\theta_2^2}}. \tag{8}$$

Hence, although the Wiener processes are independent, the correlation is non-zero. We can explain this approach as follows: in the economy, all information which can arise have impact on both assets. Thus, if one of the assets changes in the price, the other must do the same. However, we cannot observe any dependency in the direction of the price increments. The results of information arrivals are totally different.

If the two Wiener processes are correlated and we denote the correlation of ϵ_1 and ϵ_2 as ρ, we can formulate the correlation of VG processes as follows (Model 2):

$$\texttt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = \frac{\theta_1\theta_2\nu + \vartheta_1\vartheta_2\rho}{\sqrt{\vartheta_1^2 + \nu\theta_1^2}\sqrt{\vartheta_2^2 + \nu\theta_2^2}}. \tag{9}$$

Similarly to Model 1, we suppose that each information arrival results into a price change of both assets. However, we can observe either positive or negative dependency in the direction of price movements. By contrast, we do not suppose that some event can have no impact on some asset price – we always observe distinct effect, even if it can be very small in size. Hence, the global environment influences prices of all assets.

If the two Wiener processes are identical, we have a special case of (9) for $\rho = 1$.

Group B

Suppose independent gamma processes, so that we can rewrite (6) as follows:

$$\texttt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = \frac{0}{\sqrt{\vartheta_1^2 + \nu_1\theta_1^2}\sqrt{\vartheta_2^2 + \nu_2\theta_2^2}}. \tag{10}$$

Thus, the correlation of VG processes is zero irrespectively of the dependency between Wiener process.

Suppose correlated Wiener processes (Model 3). Although the correlation among Wiener processes is nonzero, from the standard time aspect, in fact they develop in random gamma time. Since respective gamma times are independent, the correlation of VG process should be also zero. We can interpret such case as follows. There can happen events of two types – with price impact either on the first asset or the second one. Sometimes, both information can arrive at the same time. However, in average, there is no correlation of event times. Although the directions of price increment are positively (or negatively) correlated (ignoring the exact time we observe them), the prices change at the same time moments only scarcely.

Group C

Suppose that we have two correlated gamma processes. This assumption can lead to various formulations. Now, we will analyze two independent Wiener processes (Model 4). Within this approach, we usually proceed in such a way that each gamma process is decomposed into two distinct gamma processes,[4] one for global information arrival (identical for both processes), the other as an idiosyncratic component (independent among each other). Define such gamma process X as follows:

$$X = G + I, \;\; G \in \mathcal{G}[a/\nu, \nu], \; I \in \mathcal{G}[(1-a)/\nu, \nu] \;\;\Rightarrow\;\; X \in \mathcal{G}[1/\nu, \nu]. \tag{11}$$

It means, that the correlation coefficient can be calculated due to:

$$\mathtt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = a\frac{\theta_1\theta_2\nu}{\sqrt{\vartheta_1^2 + \nu\theta_1^2}\sqrt{\vartheta_2^2 + \nu\theta_2^2}}. \tag{12}$$

The decomposition of gamma processes give us the nature of economic interpretation: there can happen events of three types – general, say macroeconomical, which affect prices of both (all) assets, and unique, which are relevant only for particular company (its stock or bond price).

Group D

Totally different assumption lies behind the last group of models. We do not decompose the VG process into two independent ingredients, random gamma time and Wiener process, but we would like to perceive the dependency of the overall processes. We can therefore proceed due to the VG copula model. Since this approach requires introducing of the copula theory, we do not analyze it here.

Graphical illustration

In order to allow better understanding of the dependency variants of VG processes we provide graphical illustration of Models 1 to 4 (Figure 1). In general, we suppose the following parameters: $\theta_1 = -0.03$, $\theta_2 = -0.05$, $\vartheta_1 = 0.20$, $\vartheta_2 = 0.25$ and $\nu_1 = \nu_2 = 0.1$.

3. FX rate sensitive portfolio evolution

In this section we suppose a case of financial institution with a portfolio sensitive to the evolution of foreign currency exchange rates, in particular 60% of the portfolio is

[4] The sum of two gamma processes is always a gamma process.

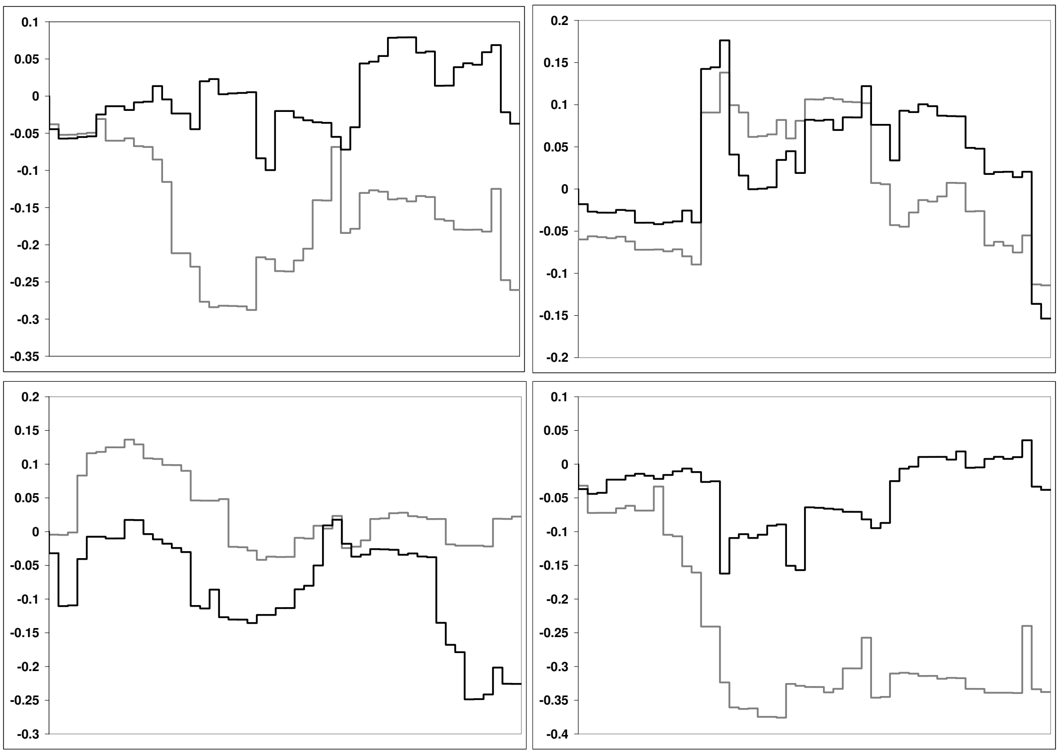

Figure 1: Evolution of two possibly dependent VG processes (Model 1 to Model 4)

sensitive to EUR/CZK, 15% to GBP/CZK, and 25% to USD/CZK FX rate. We suppose the same data set as in Tichý [15],[5] see Figure 2 for the evolution and Table 2 for respective parameters of daily returns on per annum basis. Moreover, the correlation of currency pairs are as follows: $\mathtt{cor}(EUR/CZK, GBP/CZK) \equiv \rho_A = 0.53$, $\mathtt{cor}(EUR/CZK, USD/CZK) \equiv \rho_B = 0.40$, and $\mathtt{cor}(GBP/CZK, USD/CZK) \equiv \rho_C = 0.67$.

Table 2: Basic parameters of exchange rate returns (p.a.)

τ	*EUR/CZK*				*GBP/CZK*				*USD/CZK*			
days	μ	σ	*skew*	*kurt*	μ	σ	*skew*	*kurt*	μ	σ	*skew*	*kurt*
1	-0.036	0.05	-0.24	7.28	-0.05	0.08	-0.42	5.24	-0.07	0.11	-0.14	3.83

In Tichý [15] there were suggested parameters of VG processes to match the empirical distribution of FX returns as closely as possible, see Table 3 for details. Unfortunately, these values do not allow us to model the dependency of particular currency pairs evolution. In order to do that we can apply one of the approaches suggested above.

The most intuitive approach is, probably, Model 2 with a correlation coefficient given by:

$$\mathtt{cor}[\mathcal{VG}_1, \mathcal{VG}_2] = \frac{\theta_1\theta_2\nu + \vartheta_1\vartheta_2\rho}{\sqrt{\vartheta_1^2 + \nu\theta_1^2}\sqrt{\vartheta_2^2 + \nu\theta_2^2}}.$$

[5] The time series consist of 1700 data – official exchange rates as published by ČNB (Czech National Bank) over the period starting in January, 2000 up to September, 2006.

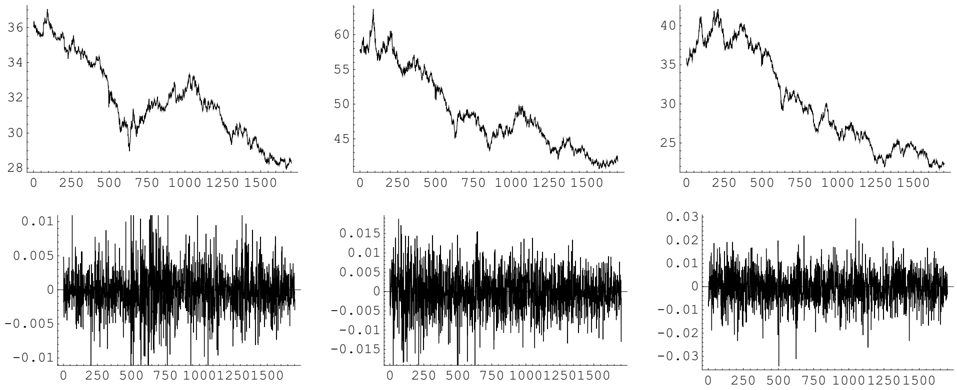

Figure 2: Daily evolution of EUR, GBP, and USD exchange rates and the continuous returns. Reproduced from [15].

Table 3: Parameters of $\mathcal{VG}(\theta, \vartheta, \nu)$

FX rate	θ	ϑ	ν
EUR	-0.0002	0.0034	1.4152
GBP	-0.00006	0.0072	0.48
USD	-0.0034	0.0095	0.37

We see that in the equation, there is only one ν, because of identical gamma processes. The first step is therefore to recalculate the parameters of VG processes to receive identical ν. Since we cannot assume, that we will be able to match the moments of the distribution exactly, it is more suitable to apply the least square approach here. The next step is to calculate particular parameters of $\rho^{\mathcal{W}}$, i.e. correlations of Wiener processes, to match the empirically observed correlation of VG processes. The results are summarized in Table 4. We see that the correlation of Wiener processes is identical to the task correlation of overall VG processes, except the sign, which depends on the combination of θ_i and θ_j.

Table 4: Parameters of correlation

FX rate pair	$\rho_{i,j}^{\mathcal{VG}}$	$\rho_{i,j}^{\mathcal{W}}$
EUR&GBP	0.53	-0.53
EUR&USD	0.40	-0.40
GBP&USD	0.67	0.67

Now, we can run several thousands of independent scenarios of portfolio evolution and calculate the characteristics of its returns. We proceed due to:

$$\tilde{x}_i = \mu_i \tau + \theta_i g_\tau + \vartheta_i \sqrt{g_\tau} \varepsilon_i - \theta_i \tau.$$

The results are summarized in Table 5. It is obvious, that ignoring the dependency we would get more skewed distribution of the portfolio returns with higher peak. By contrast, the risk of the portfolio measured by the standard deviation would be underestimated.

Table 5: Portfolio characteristics

portfolio	*Parameters*			
type	μ	σ	*skew*	*kurt*
independent	1.7×10^{-4}	25.5×10^{-4}	-0.36	5.47
dependent	1.2×10^{-4}	34.2×10^{-4}	-0.17	5.28

Conclusions

In this paper we have tried to clarify basic approaches to the modeling of asset prices or their returns via VG model, including the dependency. We have presented several distinct approaches to formulate the correlation factor of VG returns. We have also included potential economic interpretations so that it is more clear for which cases particular models are useful. For example, models in Group A are useful, if the asset prices jumps always together. By contrast, models in Group C can be useful if there are two sources of information arrivals, given by global environment and idiosyncratic ones.

Finally, we have applied Model 2 in order to estimate the parameters of currency portfolio returns. We first recalculate the parameters of VG processes to get identical parameter ν. Next, we complete the model by matching it with empirical correlation of currency pairs. Finally, we have run a simulation of portfolio evolution to calculate the parameters of portfolio distribution. This result can be used e.g. for risk assessment.

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