Binomial model and transaction costs

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Abstract

In this paper we study lattice models in presence of transaction costs. Transaction costs can be modeled as a fixed charge or a fee proportional to the price of traded assets. Here we suppose only proportional transaction costs. Firstly, we derive the simple binomial model. Secondly, we impose proportional symmetric cost on trading with the risky asset. We develop basic equations for single-period model and also a general one for the intermediate interval of the multi-period model. In this paper we suppose initial zero position and the need of physical delivery at the terminal time. We compare the results to the Boyle and Vorst model of zero initial transaction cost which clearly underestimate the price. However, we show that the absolute amount of the replication capital invested into the risky asset stays the same. We also provide the effect of portfolio model which can be used to explain some frictions at the real market.

Keywords

Lattice model, binomial model, option, transaction cost

1 Introduction

The binomial model was originally presented in 1979 by Cox, Ross and Rubinstein [9] (henceforth the CRR model) as a simplification to the more complicated Black and Scholes model [6]. The big advantage of the CRR model is that it allows to valuate not only European calls and puts but also many types of more or less exotic payoffs and, what is more important, also American options.

Both models have been initially set into idealized market conditions – constant (deterministic) parameters of drift, diffusion and riskless rate, unconstrained liquidity, no transaction costs. However, the CRR model is very intuitive so we can easy relax many of these assumptions and provide more general (or relevant) price of an option.

In this paper we handle first of all with plain vanilla option pricing within binomial setting and under consideration of transaction costs. Although the transaction costs related to executed buy or sell orders in hedging of huge books of options does not usually play a significant role as we will see later, we should realize the importance of executed transactions in case of hedging of few options on low liquidity markets.

In general, we can distinguish following types of transaction costs:

- strictly proportional to the price of the traded asset

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- fixed charge due to the number of items traded
- and various combinations of above.

The most important models incorporating transaction costs are the "continuous-time" Leland model [19] and the binomial model of Boyle and Vorst [8] (BV model). Both of them are valid only for convex payoffs with some other restrictions on input data. Next, in both models only symmetric transaction costs are supposed.

The Leland model reformulates the Black and Scholes model [6] by introducing of an augmented volatility, $\sigma_A = \sigma\sqrt{1 + A_L}$, where $A_L = \sqrt{\frac{2}{\pi}}\frac{\kappa}{\sigma\sqrt{\Delta t}}$ and κ indicates transaction costs, Δt is the trading interval.[2] During later years the same author studied carefully also the effect of capital gains taxes on the optimal number of portfolio transactions, which could also change the fair option price, see e.g. [20].

By contrast, the Boyle and Vorst model is discrete-time model. However, they provided also a closed-form approximation which resulted into Black and Scholes-type model with volatility given as $\sigma_A = \sigma\sqrt{1 + A_{BV}}$, where $A_{BV} = \frac{\kappa}{\sigma\sqrt{\Delta t}}$. Since $\sqrt{\frac{2}{\pi}} < 1$, the Boyle and Vorst model should provide slightly higher transaction spread comparing with the one of Leland.

The world of asymmetric transaction costs was initially examined by Stettner [29] and Rutkowski [27]. This assumption was also relaxed e.g. by Palmer [23], who, moreover, have provided conditions under which the results of Boyle and Vorst do (not) hold. Furthermore, he also slightly extended the superreplicating model of Bensaid *et al.* [5]. Very interesting is also another extension of the CRR model provided by Bakstein [2], who closely connected the transaction costs with the liquidity of the market, see also Bakstein and Howison [3]. Some extensions were also provided by Avellaneda and Parás [1]. Recently, Melnikov and Petrachenko [21] have extended the study also for different rates on riskless borrowing and lending. However, Roux and Zastawniak [24] have specified when this model can lead to arbitrage opportunity. Similarly to corrections of other related papers, the reason is that a superreplicating portfolio can be, under transaction costs, cheaper comparing with the one which perfectly replicate a derivative asset, see also [23] and [1].

From a vast number of models considering primarily the continuous-time world, state the following. Simple model, based on superreplicating portfolio (thus, dominating of the payoff), is the one of Soner *et al.* [28]. However, all such papers concluded, that in the limiting case the cheapest strategy should be trivial – buy (sell) and hold. Completely different approach was chosen by Hodges and Neuberger [14] who introduced the investor's preferences. Both approaches were further developed in many other papers and also combined by Barles and Soner [4].

In this paper we deal only with the most simple preference free model of symmetric transaction costs as was supposed by Boyle and Vorst. However, we add to the model the cost on initial setting of the replicating portfolio. We proceed as follows. In the following section we briefly describe and derive the single period binomial model. Next, we study the effect of proportional transaction costs on the initial replication equation to be able to derive the model in different settings of single period model, long position model, short position model, Boyle and Vorst setting and multi-period model. Finally, we analyze the effect of the portfolio model.

[2] Some argumentations on which this model were built were rather on a heuristic level. In some manner it was reconsidered by Kabanov and Safarian [16] who also developed the procedure to calculate the limiting model error.

2 Binomial model of CRR

Under classical single-period binomial model (see Cox *et al.* [9]) it is supposed that knowing the present price the price of any risky asset can take two values in the next time moment. Consider one risky asset, say stock $\mathcal{S}(t)$, with price at time zero $\mathcal{S}_0$ and one riskless asset $\mathcal{B}(t)$, which gains riskless rate r, i.e $\mathcal{B}(1) = B(0) \cdot (1 + r)$.

Under simple binomial model we suppose, that there is one source of uncertainty, say $\mathcal{Z}$, which value at time one can be described by

$$\mathcal{Z} = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p. \end{cases} \tag{1}$$

It implies that the stock price at time one can be written as

$$\mathcal{S}_1 = \mathcal{S}_1(\mathcal{Z}) = \begin{cases} \mathcal{S}_0 \cdot u & \text{with probability } p \\ \mathcal{S}_0 \cdot d & \text{with probability } 1-p. \end{cases} \tag{2}$$

The parameters u and d in equation (2) can be interpreted as indices of *up* or *down* movements in the price. Alternatively, we can formulate the (discrete-time) returns μ of the asset price conditionally on $\mathcal{Z}$ as

$$\mu(\mathcal{Z}) = \begin{cases} 1-u & \text{with probability } p \\ 1-d & \text{with probability } 1-p. \end{cases} \tag{3}$$

Here, $p \geq 0$ is the true market probability from a set of such probabilities $\mathbb{P}$. Note, that if u is the index of an up movement, it is higher than d and to the model make sense, the riskless return (index of riskless change $R = 1 + r$ to be more exact) must lie between u and d indices. Hence, the basic market condition is

$$d \leqq 1 + r \leqq u. \tag{4}$$

Suppose for a moment that (4) does not hold – for example, $1 + r \geq u$. This means that whichever the probabilities of *up* and *down* movements are, the return of the risky asset is no longer higher than the return of the riskless asset. This implies that under standard assumption of risk aversion, no one will intend to invest in the risky asset.

The standard approach to price any derivative asset f is based on the no-arbitrage condition. Hence, we are trying to construct the replication portfolio $\mathcal{H}$ which will replicate the value of f exactly (or perfectly) for all states of the world. For the model (1), the following equality must hold with probability one:

$$P\left[f_1(\mathcal{Z}) = \mathcal{H}_1(\mathcal{Z})\right] = 1. \tag{5}$$

Thus, the value of the replicating portfolio $\mathcal{H}$ must be equal to the value of f whichever the value of $\mathcal{Z}$ will be. Hence, the portfolio Π, consisting of long position in f and short position in $\mathcal{H}$ (or vice versa), will have a deterministic value at time one:

$$t = 1: \ f_1 - \mathcal{H}_1 = 0. \tag{6}$$

Since the portfolio is riskless it must earn riskless return r. Clearly, present (or future) value of zero must be always zero. Thus, it also holds that

$$t = 0: \ f_0 - \mathcal{H}_0 = 0. \tag{7}$$

Consider now the European option f, whose payoff at maturity is given by $\Psi(\mathcal{Z})$. Thus, we have a model with one source of uncertainty ($\mathcal{Z}$) and two possible states in the future at one side and $n+1$ independent assets (i.e. n ($n=1$) independent risky asset $\mathcal{S}$ + one riskless $\mathcal{B}$) on the other side. This indicates, that the market is complete, we can find the unique risk-neutral probabilities $\mathbb{Q}$ to get the risk-neutral price of the option f by appraising the unique replicating portfolio $\mathcal{H}$.

Denote the structure of the replicating portfolio by $\mathcal{H}(x,y)$, where x indicates the amount invested into $\mathcal{B}$ and y into $\mathcal{S}$, both at time zero. Hence

$$t=0: \ \mathcal{H}(x,y) = x\mathcal{B} + y\mathcal{S}_0 \tag{8}$$

and

$$t=1: \ \begin{cases} \mathcal{Z}(1)=u & \rightarrow \ \mathcal{H}(x,y) = x\mathcal{B}(1+r) + y\mathcal{S}_0 u \\ \mathcal{Z}(1)=d & \rightarrow \ \mathcal{H}(x,y) = x\mathcal{B}(1+r) + y\mathcal{S}_0 d. \end{cases} \tag{9}$$

We have stated above, that we should be looking for such $\mathcal{H}$ that its time one value will be equal to the option payoff $\Psi_{\mathcal{T}}(\mathcal{Z})$ regardless the state $\mathcal{Z}$. Therefore,

$$t=1: \ \begin{cases} \mathcal{Z}(1)=u & \rightarrow \ \Psi(u) = x\mathcal{B}(1+r) + y\mathcal{S}_0 u \\ \mathcal{Z}(1)=d & \rightarrow \ \Psi(d) = x\mathcal{B}(1+r) + y\mathcal{S}_0 d. \end{cases} \tag{10}$$

Note, that the maturity time is the only moment when we can uniquely determine the financial option value respecting its payoff, $f_{\mathcal{T}}(\mathcal{Z}) = \Psi_{\mathcal{T}}(\mathcal{Z})$, without considering any other conditions. It means that (10) results into two equations with two unknowns x and y. Setting $\mathcal{B}=1$ and solving gets:

$$x = \frac{\Psi(d)u - \Psi(u)d}{(1+r)(u-d)}, \tag{11}$$

$$y = \frac{\Psi(u) - \Psi(d)}{S(u-d)}. \tag{12}$$

The no-arbitrage condition should imply that if (10) holds then from (8):

$$t=0: \ f_0 = x\mathcal{B} + y\mathcal{S}_0. \tag{13}$$

Thus, putting x and y from (11) and (12) into (13) we get

$$f_0 = \frac{1}{1+r}\left[q\Psi(u) + (1-q)\Psi(d)\right]. \tag{14}$$

Here,

$$q = \frac{(1+r)-d}{u-d} \tag{15}$$

can be interpreted as the risk-neutral probability of going up (u) and $(1-q)$ as the risk-neutral probability of going down (d). Thus the risk-neutral probability space is given by

$$\mathbb{Q} = \left\{P[\mathcal{Z}=u]=q, \ P[\mathcal{Z}=d]=1-q\right\}. \tag{16}$$

Alternatively, respecting the risk-neutral world, we can make the average value of $\mathcal{Z}$ to be riskless, thus

$$(1-q)d + qu = 1+r. \tag{17}$$

The extension of the single-period binomial model into the $n-$period model is straightforward. The risky asset price evolutes according to (2) rewritten into n-period model

$$\mathcal{S}_n = \mathcal{S}_0 \cdot \prod_{k}^{n} \mathcal{Z}_k. \tag{18}$$

Similarly, the riskless asset evolution is given by $\mathcal{B}(n) = B(0) \cdot (1+r)^n$.

Knowing the solution of (8) and (9) and applying the backward recursive procedure, we are still able to recover the option value at time t on the basis of time $t+1$ values. Thus, (14) changes into

$$f_t(\mathcal{S}_t) = \frac{1}{1+r} \cdot \left[q f_{t+1}(\mathcal{S}_t u) + (1-q) f_{t+1}(\mathcal{S}_t d) \right]. \tag{19}$$

Taking these results into account, we can formulate a time zero value of an option with general (European) payoff $\Psi(\mathcal{S}_\mathcal{T})$ as

$$f_0 = \frac{1}{(1+r)^n} \cdot \sum_{j=0}^{n} \mathrm{Co}\tbinom{n}{j} q^j (1-q)^{n-j} \Psi(\mathcal{S} u^j d^{n-j}). \tag{20}$$

3 Transaction costs

Transaction costs imposed on trades in the economy are usually modeled by the *bid/ask* spread.[3] Hence, prices relevant when buying the asset ($\mathcal{S}^{ask}$) are strictly higher than price at which we can sold the same asset ($\mathcal{S}^{bid}$) at the same time.

In the first subsection we consider the case of symmetric transaction costs within single period model with general payoff. Subsequently, we analyze more simple cases of European plain vanilla call with terminal y either one or zero. This allows us to simplify the model substantially. We also study the case of shorted call option which is slightly similar. After some modifications, these procedures and/or results can be used to developed the replicating equations and option pricing formulas for many derivatives with closely related payoffs.

Each model of this section suppose zero initial position, with one exception. For comparison reasons we provide also Boyle and Vorst model which does not take such fact into account.

3.1 Case 1 – single period, symmetric κ, general payoff

In order to simplify the treating of transaction costs, we can suppose that at any time t we have to handle with following prices:

$$\mathcal{S}_t(1-\kappa) = \mathcal{S}_t^{bid} \leq \mathcal{S}_t \leq \mathcal{S}_t^{ask} = \mathcal{S}_t(1+\kappa), \tag{21}$$

where $\kappa \geq 0$ indicates percentage transaction costs, which are symmetric. This expression indicates that if we buy the asset $\mathcal{S}$ we must pay the *ask* price (relevant negative cash

[3] As transaction costs sensu largo can be assign also costs on the wage of individuals who monitor the rebalancing of the replicating portfolio.

flow is $\mathcal{S}_t^{ask}$) and if we sell the asset $\mathcal{S}$ we receive the *bid* price (relevant positive cash flow is $\mathcal{S}_t^{bid}$).

First, in order to construct the portfolio $\mathcal{H}$ replicating the option payoff, we must specify the initial holding and whether the option results into cash or physical delivery. Here, we will suppose that the initial position is $\mathcal{H}_0(x = 0, y = 0)$ and the option must result into physical delivery of the underlying asset. Hereafter, we will call the model TTC model (total transaction costs).

It is also useful here to start distinguish among $\mathcal{H}_0(x_0, y_0)$ (the position in the riskless asset $\mathcal{B}$ and the risky asset $\mathcal{S}$ at the beginning), $\mathcal{H}_1(x_1, y_1)$ (the position which is set at time zero to replicate the option at time one, i.e. it is predictable at time zero), and $\mathcal{H}_T(x_T, y_T)$ which indicates the delivery. If the option is exercised we get $\mathcal{H}_T(-\mathcal{K}, 1)$, if it is not the case, then $\mathcal{H}_T(0, 0)$.

The transaction costs on initial setting of the portfolio implies that the value of the replicated option is

$$t = 0: \; f_0 = \mathcal{H}_0(x_0 = f_0, y_0 = 0) = x_0\mathcal{B} + y_0\mathcal{S}_0. \tag{22}$$

and the value of the replication portfolio at time zero (8), which should predicts the value of the option at time one changes into

$$t = 0: \; \mathcal{H}_1(x_1, y_1) = H_0(x_0, y_0) - |y_1 - y_0|\mathcal{S}_0\kappa = x_1\mathcal{B} + y_1\mathcal{S}_0 - |y_1 - y_0|\mathcal{S}_0\kappa. \tag{23}$$

In the last term of equation (23) we deduce the transaction costs given by the purchase (sell) of the asset $\mathcal{S}$. Clearly, it reduces the amount of money intended for riskless position. Similarly, the system (10) changes into

$$t = 1: \begin{cases} \mathcal{Z}(1) = u \;\; \rightarrow \;\; \Psi(u) + |y_T - y_1|\mathcal{S}u\kappa = (x_1\mathcal{B} - |y_1 - y_0|\mathcal{S}_0\kappa)(1 + r) + y_1\mathcal{S}_0 u \\ \mathcal{Z}(1) = d \;\; \rightarrow \;\; \Psi(d) + |y_T - y_1|\mathcal{S}d\kappa = (x_1\mathcal{B} - |y_1 - y_0|\mathcal{S}_0\kappa)(1 + r) + y_1\mathcal{S}_0 d. \end{cases} \tag{24}$$

The second term on the left hand side of the equality indicates that the final value of the replication portfolio must be such that it will be sufficient to make a physical delivery. Hence, it is equal to the option payoff plus the cost on executing of terminal rebalancing transaction. Note that this formulation is general enough to catch almost all European payoffs (calls, puts, barriers).

3.2 Case 2 – single period, symmetric κ, long vanilla call

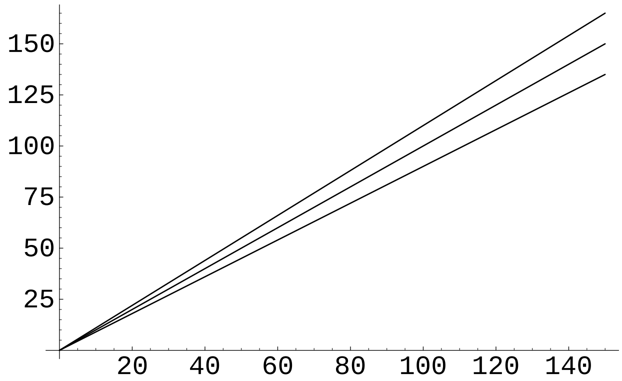
We will show now the solution of equations (23) and (24) considering the (European) plain vanilla call option. This is a financial derivative which gives the owner the right to buy the underlying asset at maturity by paying the prespecified exercise price $\mathcal{K}$. Therefore, the payoff function of this option is given by $\Psi_{call}^{vanilla} = (\mathcal{S}_\mathcal{T} - \mathcal{K})^+ \equiv \max(\mathcal{S}_\mathcal{T} - \mathcal{K}; 0)$. In this subsection we suppose the purchase of the option – we are trying to replicate the long position.

The equations can be simplified, since the y is strictly increasing with $\mathcal{S}$ (from zero to one) and if we know its value at maturity (which we already know in the case of the single period model) – it is either one ($\Psi(u) > 0$) or zero ($\Psi(d) = 0$). Hence

$$t = 0: \; \mathcal{H}_1(x_1, y_1) = x_1\mathcal{B} + y_1\mathcal{S}_0 - y_1\mathcal{S}_0\kappa \tag{25}$$

and

$$t = 1: \begin{cases} \mathcal{Z}(1) = u \;\; \rightarrow \;\; \Psi(u) + (1 - y_1)\mathcal{S}_0 u\kappa = (x_1\mathcal{B} - y_1\mathcal{S}_0\kappa)(1 + r) + y_1\mathcal{S}_0 u \\ \mathcal{Z}(1) = d \;\; \rightarrow \;\; \Psi(d) + y_1\mathcal{S}_0 d\kappa = (x_1\mathcal{B} - y_1\mathcal{S}_0\kappa)(1 + r) + y_1\mathcal{S}_0 d. \end{cases} \tag{26}$$



Obrázek 1: Bid/Ask spread of $\mathcal{S}$

From (26) we obtain

$$x_1 = \frac{((1+r)\kappa - d(1-\kappa))(\Psi(u) + \kappa S_0 u) + \Psi(d)(u + \kappa(u - (1+r)))}{(1+r)((1+\kappa)u - d(1-\kappa))} \tag{27}$$

and

$$y_1 = \frac{\Psi(u) - \Psi(d) + \kappa S_0 u}{S_0((1+\kappa)u - d(1-\kappa))}. \tag{28}$$

Putting x_1 and y_1 from (27) and (28) into (25) we get the call option valuation formula as follows:

$$f_0 = \frac{1}{1+r}\left[(\Psi(u) + S_0\kappa u)\frac{(1+r)(1+\kappa) - d(1-\kappa)}{(1+\kappa)u - d(1-\kappa)} + \Psi(d)\frac{(u-(1+r))(1+\kappa)}{(1+\kappa)u - d(1-\kappa)}\right]. \tag{29}$$

This can be rewritten by setting the artificial probability of an up movement q,

$$q = \frac{(1+r)(1+\kappa) - d(1-\kappa)}{(1+\kappa)u - d(1-\kappa),} \tag{30}$$

as

$$f_0 = \frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]. \tag{31}$$

The middle term in (31) indicates the (present) value of transaction costs needed to transfer the replication portfolio at maturity into the asset $\mathcal{S}$. This is given by the fact that it equals option price at time zero minus risk-neutral present value of the payoff:

$$\kappa u S_0 q(1+r)^{-1} = f_0 - \frac{1}{1+r}\left[\Psi(u)q + \Psi(d)(1-q)\right]. \tag{32}$$

Consider the one-year call option on $\mathcal{S}$ with $\mathcal{K} = 100, \sigma = 0.25$, and $r = 0.05$. Suppose that $\mathcal{S}_0 \in (80; 150)$, which indicates, that $\Psi(u) > 0\ \forall \mathcal{S}_0$. Consider three different transaction costs, in particular $\kappa = 0\%$, $\kappa = 5\%$, and $\kappa = 10\%$, respectively. Figure 1 indicates the spread on the asset price, Figure 2 shows the option price. Note, that if the initial price of the underlying asset is lower than $\mathcal{K}/u$, the option cannot be exercised in any case and it does not make sense to compose the replication portfolio – moreover, its value would be negative.

Explain for example the situation if $\mathcal{S}_0 = 80$ and $\kappa = 10\%$. The underlying asset price at time $t = 1$ will be either 102.7 or 62.3. It implies the intrinsic value (option payoff) at maturity to be either $\Psi(u) = 2.72$ or $\Psi(d) = 0$. Now, we can calculate the initial option price by discounting the expected payoff including the portion of transaction cost as given by formulation (31) and we get $f_0 = 7.9$, see Table 1 and Table 2.

Tabulka 1: Price evolution I.

	underlying asset price			*call option value*		
	time	0	1	time	0	1
	1		102.7	1		2.72
state	0	80		0	?	
	-1		62.3	-1		0

Table 2 indicates the composition of the replicating portfolio $\mathcal{H}$. The riskless position is negative, which indicates riskless borrowing, and involves the costs on setting up of the initial position (purchase of the underlying asset according to $y_1 = 0.1826$, see equation (28) – this implies the risky position at time zero). Thus, it is the sum of $x_1\mathcal{B}$ and $-y_1\mathcal{S}_0\kappa$. Hence,

$$x_1\mathcal{B} - y_1\mathcal{S}_0\kappa = -10.37 - 1.83 = -12.2.$$

Putting together the riskless position and the risky position we get the total value of the portfolio at the beginning. Note however, that since $f_0 = \mathcal{H}_1(x, y) + y_1\mathcal{S}_0\kappa$ we must add the cost on initial setting of the portfolio back in order to get the fair price of the option.

Now, we move further to the next time moment, $t = 1$. Clearly, the $t = 1$ value of the riskless position does not depend on the state of the world $\mathcal{Z}$. The value of the risky part in the middle pannel of Table 2 is before final rebalancing – it is the product of the delta calculated at time zero and the time $t = 1$ asset price, $y_1 \cdot \mathcal{S}_1(\mathcal{Z}) = 0.1826 \cdot \mathcal{S}_1(\mathcal{Z})$. Again, putting together the value of the risky and riskless part we get the value of the portfolio (before final rebalancing).

Tabulka 2: Effect of transaction costs I.

riskless position				*risky position*				*total value*		
time	0	1	($\mathcal{T}$)	time	0	1	($\mathcal{T}$)	time	0	1
1		-12.8	(-100)	1		23.45	(102.7)	1		10.65
0	-12.2			0	18.26			0	6.07	
-1		-12.8	(0)	-1		14.22	(0)	-1		1.42

Recall now, that the option must lead to physical delivery. Therefore, at maturity time the owner of this financial derivative will own the whole share of the underlying asset and due the cash equivalent to the exercise price or will have zero position. Thus, we must execute final rebalancing and set the $y_{\mathcal{T}}$ either to one or zero. Values of all positions (riskless, risky, total) after terminal rebalancing are given in Table 2 in brackets.

Suppose that $\mathcal{Z} = u$. Thus, $\mathcal{S} = 102.7$, $y_{\mathcal{T}} = 1$. It follows that

$$(y_{\mathcal{T}} - y_1)\mathcal{S}u\kappa = (1 - 0.1826)102.7 \cdot 0.1 = 7.93.$$

Clearly, deducing this quantity from the relevant number in the last panel of Table 2 we get the payoff value of 2.72.

Furthermore, we can examine if the present value of transaction cost given by the term (32) is really equal to the difference between the initial option price and the present value of the expected payoff. Hence,

$$\kappa \mathcal{S}_0 uq/1 + r = 6.24$$

which is clearly the difference between

$$f_0 = 7.9$$

and

$$\text{PV}\left(\mathbb{E}[\Psi(\mathcal{Z})]\right) = \left(q\Psi(u) + (1-q)\Psi(d)\right)/(1+r) = 1.65.$$

Here we can see, that final transferring of the replicating portfolio either into the holding of the underlying asset (purchase of another fraction of $\mathcal{S}$ if $\mathcal{Z}(1) = u$) or into zero position in both assets (selling all shares of $\mathcal{S}$ if $\mathcal{Z}(1) = d$) changes substantially the value of the portfolio. We have also examined that this portfolio really allow us the riskless replication of the option. In Table 3 and 4 we can see the effect of transaction costs on the setting up of the replication portfolio as given by the initial underlying asset price $\mathcal{S}_0 = 100$.

Tabulka 3: Price evolution II.

	underlying asset price			*call option value*		
	time	0	1	time	0	1
	1		128.4	1		28.4
state	0	100		0	?	
	-1		77.9	-1		0

Tabulka 4: Effect of transaction costs II.

riskless position				*risky position*				*total value*		
time	0	1	($\mathcal{T}$)	time	0	1	($\mathcal{T}$)	time	0	1
1		-40.63	(-100)	1		74.43	(128.4)	1		33.8
0	-38.7			0	58			0	19.27	
-1		-40.63	(0)	-1		45.14	(0)	-1		4.5

3.3 Case 3 – single period, symmetric κ, short vanilla call

Although in practice more common task is to replicate the long position, which is given by the need to hedge the risk of short positions, it is also important to know the fair price for the seller of the option – thus the short position.

The main difference appears at maturity time – the payoff is either negative ($\Psi(u) < 0$) or zero ($\Psi(d) = 0$). Next, the y is inversely related to the underlying asset price which turns several equations. Moreover, at maturity it is equal either minus one or zero. However, before maturity time, it can break these bounds under some circumstances. Therefore, the system of equations bellow will not hold for deeply out- or in-the-money options.

The first equation looks like follows:[4]

$$t = 0: \ \mathcal{H}_1(x_1, y_1) = x_1\mathcal{B} + y_1\mathcal{S}_0 + y_1\mathcal{S}_0\kappa. \tag{33}$$

Then

$$t = 1: \begin{cases} \mathcal{Z}(1) = u & \rightarrow \quad \Psi(u) + (1+y_1)\mathcal{S}_0 u\kappa = (x_1\mathcal{B} + y_1\mathcal{S}_0\kappa)(1+r) + y_1\mathcal{S}_0 u \\ \mathcal{Z}(1) = d & \rightarrow \quad \Psi(d) - y_1\mathcal{S}_0 d\kappa = (x_1\mathcal{B} + y_1\mathcal{S}_0\kappa)(1+r) + y_1\mathcal{S}_0 d. \end{cases} \tag{34}$$

From (34) we obtain

$$x_1 = \frac{-((1+r)\kappa + d(1+\kappa))(\Psi(u) + \kappa S_0 u) + \Psi(d)(u + \kappa((1+r) - u))}{(1+r)(u(1-\kappa) - d(1+\kappa))} \tag{35}$$

and

$$y_1 = \frac{\Psi(u) - \Psi(d) + \kappa S_0 u}{S_0((u(1-\kappa) - d(1+\kappa))}. \tag{36}$$

Putting x_1 and y_1 from (35) and (36) into (33) we get the call option valuation formula as follows:

$$f_0 = \frac{1}{1+r}\left[(\Psi(u) + S_0\kappa u)\frac{(1+r)(1-\kappa) - d(1+\kappa)}{u(1-\kappa) - d(1+\kappa)} + \Psi(d)\frac{(1-\kappa)(u - (1+r)}{u(1-\kappa) - d(1+\kappa)}\right]. \tag{37}$$

This can be rewritten by setting the artificial probability of an up movement

$$q = \frac{(1+r)(1-\kappa) - d(1+\kappa)}{u(1-\kappa) - d(1+\kappa)} \tag{38}$$

as

$$f_0 = \frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]. \tag{39}$$

Figure 2 shows the value of short European option including transaction costs on riskless replication as based on inputs of preceding subsection. The middle line is the value according to the CRR model. The respective lines of bid and ask values indicates the spread given by the level of transaction costs. Apparently, it is more wide than for the underlying asset.

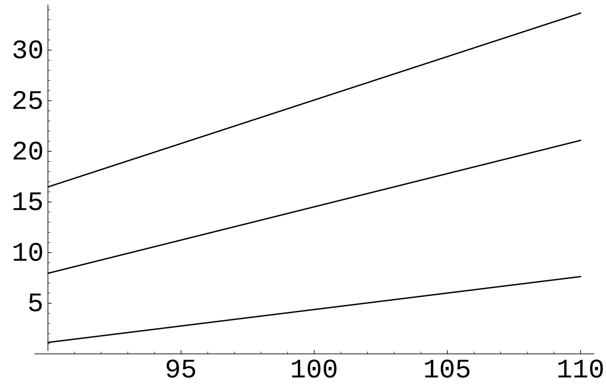
3.4 Case 4 – Boyle and Vorst model, symmetric κ

Boyle and Vorst made one important assumption which significantly simplify the system of equations (although it is apparent mainly for multi-period models) and can substantially reduce the total amount of transaction costs. They suppose that the market participant do not need to execute the first transaction – the setting up of the replicating portfolio at time zero.

Hence, the first equation is of the following form:

$$t = 0: \ f_0 = \mathcal{H}_1(x_1, y_1) = x_1\mathcal{B} + y_1\mathcal{S}_0. \tag{40}$$

[4] Since parameter y is negative, in order to deduce transaction costs we must put here the minus sign.



Obrázek 2: Bid/Ask spread of call option

This also slightly change the system (24) into the subsequent form:

$$t = 1: \begin{cases} \mathcal{Z}(1) = u \quad \rightarrow \quad \Psi(u) + |y_T - y_1|\mathcal{S}u\kappa = x_1\mathcal{B}(1+r) + y_1\mathcal{S}_0 u \\ \mathcal{Z}(1) = d \quad \rightarrow \quad \Psi(d) + |y_T - y_1|\mathcal{S}d\kappa = x_1\mathcal{B}(1+r) + y_1\mathcal{S}_0 d. \end{cases} \tag{41}$$

Supposing long plain vanilla call and under some mild restrictions on input data, we can again rewrite this system into the more simple form:

$$t = 1: \begin{cases} \mathcal{Z}(1) = u \quad \rightarrow \quad \Psi(u) + (1 - y_1)\mathcal{S}u\kappa = x_1\mathcal{B}(1+r) + y_1\mathcal{S}_0 u \\ \mathcal{Z}(1) = d \quad \rightarrow \quad \Psi(d) + y_1\mathcal{S}d\kappa = x_1\mathcal{B}(1+r) + y_1\mathcal{S}_0 d. \end{cases} \tag{42}$$

Once again, we can obtain

$$x_1 = \frac{(-d(1-\kappa))(\Psi(u) + \kappa S_0 u) + \Psi(d)u(1+\kappa)}{(1+r)((1+\kappa)u - d(1-\kappa))} \tag{43}$$

and

$$y_1 = \frac{\Psi(u) - \Psi(d) + \kappa S_0 u}{S_0((1+\kappa)u - d(1-\kappa))} \tag{44}$$

which, setting

$$q = \frac{1 + r - d(1-\kappa)}{(1+\kappa)u - d(1-\kappa)}, \tag{45}$$

results again into

$$f_0 = \frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]. \tag{46}$$

3.5 Case 5 – multi-period model, symmetric κ

Pricing of the option and hedging of its payoff is usually done in more than only one step. In this subsection we will look more closely on the multi-period model.

Suppose, that knowing the initial holding at time t, $\mathcal{H}_t(x_t, y_t)$, we are able to predict the holding for the time $t+1$, $\mathcal{H}_{t+1}(x_{t+1}, y_{t+1})$, to the portfolio be worth exactly the same amount as the option plus expected cost on transferring positions in both asset. Thus, rewriting (24) we get

$$t+1: \begin{cases} \mathcal{Z}(t+1) = u \quad \rightarrow \quad f_{t+1}(u) + |y_{t+1} - y_t|\mathcal{S}_t u\kappa = x_{t+1}\mathcal{B}(1+r) + y_{t+1}\mathcal{S}_t u \\ \mathcal{Z}(t+1) = d \quad \rightarrow \quad f_{t+1}(d) + |y_{t+1} - y_t|\mathcal{S}_t d\kappa = x_{t+1}\mathcal{B}(1+r) + y_{t+1}\mathcal{S}_t d. \end{cases} \tag{47}$$

Hence, we can apply the standard backward recursive procedure to get the initial option price starting with the terminal payoff up to the time one. Subsequently, to get the time zero value, we must also take into account the cost on setting up of the initial transaction, see equation 23. Note, that we cannot utilize all results of preceding subsections since the intermediate values of y will probably differ from one/zero.

Suppose now that we intend to replicate and price long position in plain vanilla call within three steps. The input data are analogous to the cases above. The underlying asset price evolution is clear from Table 5.

Tabulka 5: Three period model – evolution of $\mathcal{S}$

		underlying asset price			
	time	0	1	2	3
	3				154.19
	2			133.47	
	1		115.52		115.53
state	0	100		100	
	-1		86.56		86.56
	-2			74.93	
	-3				64.86

Recall, that the only one time at which we know the option price exactly is its maturity. Therefore we must start our calculation at time $t = 3$. As a first step, we fill in the option price, see the last column of Table 8. There is also no doubt about the value of parameter y (option *delta*) at maturity – it is either one (states 3 and 1) or zero (states -1 and -3), see Table 6.

Tabulka 6: Three period model – parameter y

		parameter y			
	time	0	1	2	3
	3				1
	2			1	
	1		0.72		1
state	0	0.58		0.55	
	-1		0.38		0
	-2			0.23	
	-3				0

Subsequently, we proceed to time $t = 2$. At first, we set up the relevant equations, see e.g. general equations of Boyle and Vorst model (40) and (41). Solving these equations we get right values of x, y and f at nodes[5] $(2, 2)$, $(0, 2)$ and $(-2, 2)$. Note, that we can utilize the final results of subsection 3.4 only to get the values at $(0, 2)$. Clearly, calculating first the artificial probability q according to (45) and putting it into (46) we get the same $f(0, 2) = 12.87$.

[5]Here, by node we mean the coordinates of $(state, time)$.

Very interesting is also the value of y at $(-2, 2)$ – although the option cannot be exercised in anyone of the subsequent states the "delta" is positive, which results into long holding of the underlying asset. It is apparent that this is the results of expected transaction costs.

Tabulka 7: Three period model – parameter x

		parameter x			
	time	0	1	2	3
	3				-
	2			-98.36	
	1		-56.77		-
state	0	-31.97		-42.20	
	-1		-22.55		-
	-2			-13.49	
	-3				-

Further, we proceed to time $t = 1$ and finally we get the price at the beginning ($t = 0$) applying the general equations of subsection 3.2. Since we add the costs on the initial setting of the portfolio at time zero, the value is higher than the present value of the expectation in time one. Note, that if we supposed no cost on the initial setting of the portfolio (Boyle and Vorst model), its value would be $f_0 = 20.07$.

Tabulka 8: Option value

		value of the option			
	time	0	1	2	3
	3				54.19
	2			35.11	
	1		26.78		15.53
state	0	25.85		12.87	
	-1		9.95		0
	-2			4.11	
	-3				0

4 Application on single option and portfolio of options

It is well known fact that hedging large book of options can be significantly different to hedging of single option. Hence, in this section we plan to verify the role of transaction costs in such cases.

As before, consider the following underlying asset $\mathcal{S}$ with the initial price $\mathcal{S}_0 = 100$, annual volatility of returns $\sigma = 0.25$ and the riskless rate $r = 0.05$. The portfolio Π to be replicated consists of five different call options ${}_nf_{\mathcal{K}}^{sign}$, where n indicates the amount of options in the portfolio, $\mathcal{K}$ is the exercise price and $sign$ is either " + " (long position) or " − " (short position):

$$\Pi = \left({}_2f^+_{90},\ {}_2f^+_{100},\ {}_1f^-_{95},\ {}_1f^-_{105},\ {}_1f^-_{115}\right).$$

The input data indicates, that exercised is either each option or none. Further more, the y (option delta) at maturity is either one ($2\mathcal{S} + 2\mathcal{S} - \mathcal{S} - \mathcal{S} - \mathcal{S}$) or zero and the final payoff is $\Psi(u) = 63.4$ or $\Psi(d) = 0$.

Following Table 9 includes in particular columns values of option (CRR model and TTC model) and relevant values of parameter y. In particular rows we examine single options. We also calculate the sum of all values and finally provide the portfolio model.

It is apparent that applying the replication equations on the whole portfolio of options can significantly decrease the total costs. Although the initial delta is slightly higher than the total sum, we do not need to trade so much. By contrast, it is quite surprising, that the cost are lower also to the case of CRR model.

Tabulka 9: Portfolio of options

		value of the option			
	amount	*value*	y	*value*	y
${}_2f^+_{90}$	2	14.52	0.56	25.07	0.58
${}_2f^+_{100}$	2	19.63	0.76	31.15	0.72
${}_1f^-_{95}$	1	-17.08	-0.66	-5.79	-0.69
${}_1f^-_{105}$	1	-11.96	-0.46	-2.97	-0.35
${}_1f^-_{115}$	1	-6.85	-0.26	-0.16	-0.02
sum		32.41	1.26	103.52	1.54
Π		32.41	1.26	14.23	1.69

Replication of a whole portfolio as a single derivative has also another effect. Suppose that the portfolio consist of a long and a short position in the same option. Since results of the system of replication equations for long position differs from the one for the short position, making simple sum will give us portfolio value as well as portfolio y distinct from zero. Of course, it is not surprising, that the payoff of such portfolio must be zero regardless the future price of the underlying asset. And we can conclude, that if there exists an opportunity to hedge the payoff by netting of with opposite position, the transaction costs should not have any effect on the option price. Apparently, disharmony in the volume of supply and demand can significantly extend the option bid/ask spread.

5 Conclusions

Transaction costs can play very important role in pricing, hedging and replication of financial derivatives. Although the effect on the price of large books of options can be insignificant through the netting of positions, the additional capital needed to replicate (hedge) particular asset can be important.

In this paper we have shown how to price and replicate the option payoff within a binomial model. We have presented universal equations, which are valid for general payoff functions and can be used to deduce the pricing formula. In much more detail we have studied the case of plain vanilla call. More particularly, we have derived all basic equations for single period model of long and short position in an option.

Since the assumption of zero cost on initial setting of the replicating portfolio is usually not meet in practice, we have also tried extend the more simple Boyle and Vorst model and formulate the difference in particular parameters. Table 10 shows all formulas needed to implement CRR, TTC and BV model.

Finally, we have shown the effect of the portfolio model – replicating a book of particular options on the same asset as a single derivative. This allows us to reduce the total replication cost for a huge amount.

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Summary

Binomický model a transakční náklady

V tomto článku je studován binomický model za přítomnosti transakčních nákladu, přičemž je předpokládána jejich symetričost. Nejprve je odvozen základní model binomický. Následně jsou uvaleny transakvční náklady na obchodování s rizikovým (podkladovým) aktivem a je studován vliv na soustavu replikačních rovnic včetně výsledných formulí. V tomto článku jsou zahrnuty i náklady na počáteční sestavení replikačního portfolia. Na závěr je studován efekt portfolia.

Tabulka 10: Single period model parameters (vanilla call)

parameter	CRR
$\mathcal{H}_1(x_1, y_1)$	$x_1\mathcal{B} + y_1\mathcal{S}_0$
x_1	$\frac{\Psi(d)u-\Psi(u)d}{(1+r)(u-d)}$
y_1	$\frac{\Psi(u)-\Psi(d)}{S(u-d)}$
q	$\frac{(1+r)-d}{u-d}$
f_0	$\frac{1}{1+r}\left[q\Psi(u) + (1-q)\Psi(d)\right]$
parameter	**TTC long call**
$\mathcal{H}_0(x_0, y_0)$	$x_0\mathcal{B} + y_0\mathcal{S}_0$
$\mathcal{H}_1(x_1, y_1)$	$x_1\mathcal{B} + y_1\mathcal{S}_0 - y_1\mathcal{S}\kappa$
x_1	$\frac{((1+r)\kappa-d(1-\kappa))(\Psi(u)+\kappa S_0u)+\Psi(d)(u+\kappa(u-(1+r)))}{(1+r)((1+\kappa)u-d(1-\kappa))}$
y_1	$\frac{\Psi(u)-\Psi(d)+\kappa S_0u}{S_0((1+\kappa)u-d(1-\kappa))}$
q	$\frac{(1+r)(1+\kappa)-d(1-\kappa)}{(1+\kappa)u-d(1-\kappa)}$
f_0	$\frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]$
parameter	**TTC short call**
$\mathcal{H}_0(x_0, y_0)$	$x_0\mathcal{B} + y_0\mathcal{S}_0$
$\mathcal{H}_1(x_1, y_1)$	$x_1\mathcal{B} + y_1\mathcal{S}_0 + y_1\mathcal{S}_0\kappa$
x_1	$\frac{-((1+r)\kappa+d(1+\kappa))(\Psi(u)+\kappa S_0u)+\Psi(d)(u+\kappa((1+r)-u))}{(1+r)(u(1-\kappa)-d(1+\kappa))}$
y_1	$\frac{\Psi(u)-\Psi(d)+\kappa S_0u}{S_0((u(1-\kappa)-d(1+\kappa))}$
q	$\frac{(1+r)(1-\kappa)-d(1+\kappa)}{u(1-\kappa)-d(1+\kappa)}$
f_0	$\frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]$
parameter	**B&V long call**
$\mathcal{H}_1(x_1, y_1)$	$x_1\mathcal{B} + y_1\mathcal{S}_0$
x_1	$\frac{(-d(1-\kappa))(\Psi(u)+\kappa S_0u)+\Psi(d)u(1+\kappa)}{(1+r)((1+\kappa)u-d(1-\kappa))}$
y_1	$\frac{\Psi(u)-\Psi(d)+\kappa S_0u}{S_0((1+\kappa)u-d(1-\kappa))}$
q	$\frac{(1+r)-d(1-\kappa)}{u(1+\kappa)-d(1-\kappa)}$
f_0	$\frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]$
parameter	**B&V short call**
$\mathcal{H}_1(x_1, y_1)$	$x_1\mathcal{B} + y_1\mathcal{S}_0$
x_1	$\frac{(-d(1+\kappa))(\Psi(u)+\kappa S_0u)+\Psi(d)u(1-\kappa)}{(1+r)((1-\kappa)u-d(1+\kappa))}$
y_1	$\frac{\Psi(u)-\Psi(d)+\kappa S_0u}{S_0((1-\kappa)u-d(1+\kappa))}$
q	$\frac{(1+r)-d(1+\kappa)}{u(1-\kappa)-d(1+\kappa)}$
f_0	$\frac{1}{1+r}\left[\Psi(u)q + \kappa u S_0 q + \Psi(d)(1-q)\right]$