The Black–Scholes model of evaluating of options and its modifications, a theoretical and empirical analysis

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Abstract

The Black–Scholes model is one of the key models of evaluating options in modern literature concerning finance. This article is divided into two parts. In the first part there is a short description of the model presented by Merton, Scholes and Black in 1973 together with its later modifications, the other part of the article is empirical in character and is based on WIG 20 index. The aim of the article is to show that the latest approach to the model of evaluating options gives a very good description of the reality; the model is a handy tool for risk analysis and is useful for risk management.

Key words

Black–Scholes model of evaluating options, stochastic process, student distribution, geometrical process of Brownian motion, options, martingales.

1. The Black-Scholes model and its modern modifications

In 1973 was created a model of evaluating the European option of purchasing of stocks, which did not give dividend yield. Its authors: Robert C. Merton, Myron S. Scholes and Fisher Black made a breakthrough as far as methods of evaluating financial derivatives went. There have been many variations and modifications of the Black–Scholes formula. It constitutes the basis for evaluating more and more complicated derivatives such as for example exotic options. Participants of capital market use the model for evaluating financial instruments and assessing the risk of investments. In 1979 the model was awarded the Nobel Prize in the field of economics.

1.1 The classic model of evaluating options

The model presented in 1973 has a number of theoretical assumptions; some of them are as follow [see [2], [3], [4]]:

- the market functions on the continual basis,
- short-term, free from risk interest rate is not subject to change during the validity of the option,
- interest rates have logarithmic – normal distribution with constant parameters,
- the shares are infinitely divisible,
- the prices of selling and purchasing are the same,
- the stocks do not give dividend yield during the term of validity of options,

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- the cost of the transaction is not taken into account,
- tax is not taken into account,
- there is no possibility of arbitration.

When the above conditions are met the Black–Scholes model looks like this:

$$C = S \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2)$$

$$P = X \cdot e^{-rT} \cdot N(-d_2) - S \cdot N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

where:
C – value of the European purchasing option,
P – value of the European selling option,
S – current price of the option,
X – price of carrying out the option,
r – free from risk interest rate,
T – length of the period of validity of the option (in years),
σ – standard deviation of return on stock,
N (d) – value of distribution function of standardised normal distribution for argument d.

The Black–Scholes model gives a close picture of the reality. The prices of options one can arrive at when using it are quite like the prices on the market. Analysing the form of the model can draw the following conclusions:

- the increase of the price of the stock means the increase of the value of the option,
- the longer the term of validity of the option the higher value it reaches,
- the increase of interest rate brings about the increase of the value of the option,
- standard deviation measures the risk of the stock, its increase causes the increase of the price of the option.

The above-described model was later treated as the basis for many works in the field of financial engineering. The assumptions of the first model can not always be met so it is necessary on the one hand to come up with the specification of the model and on the other hand to constantly fit it to ever changing financial market.

1.2 A chosen modification of the model of evaluating options

A close study of the behaviour of time series of the prices of stock exchange instruments results in the conclusion that some of the relevance of the original Black–Scholes model has gone. In literature dealing with finance and economics there have been marked the following properties of the stock exchange rates of return:

- processes of rates of return are not correlated,
- long-term dependence is represented by absolute values or squared rates of return,
- rates of return have Leptokurtic distribution, which has a higher maximum and heavier (thicker) tails than Gauss distribution.

In 1999 a new model of prices of stocks was created. The model was similar to the Black–Scholes model. It used the concept of the Brownian motion with fractal activity of time. [see [1] and references]. In the model the price of the stock was described by a stochastic differential equation:

$$dP_t = P_t\{\mu\, dt + \sigma\, dW(T_t)\},$$

which in turn enabled to arrive at a formula describing the price of the stock

$$P_t = P_0 \exp\left\{\mu t \quad \tfrac{\sigma^2}{2} T_t + \sigma W(T_t)\right\},$$

where μ and σ are constants whereas W is a standard Brownian motion.

Fractal activity of time in the geometrical Brownian motion found its use in the process of modelling of securities burdened with high degrees of risk.

Below a formula of evaluating the European option of selling of stocks is presented. Let us assume that S_t means the price of stock at the moment t, $\beta_t = \beta_0 e^{rt}$ means deterministic money account. On the condition that there is a lack of arbitration discounted process of the price of the stock $\{S_t / \beta_t\}$ is a martingale M_t in relation to martingale measure Q equivalent to the measure introduced by the process of the price of the stock $\{S_t\}$. Then the following equation is satisfied:

$$dM_t = M_t\{(\mu - r)\, dt + \sigma\, dW(T_t)\}$$

However, these processes are martingales only when $\mu = r$, so assuming that $\mu = r$ enables to write an equation of the price of the stock at the moment t:

$$S_t^* = S_0 \exp\left\{\left(rt - \tfrac{1}{2}\sigma^2 T_t\right) + \sigma W(T_t)\right\}$$

in relation to a martingale measure Q.

When the price of an option of purchasing of stock is marked as CC then according to the Black-Scholes formula:

$$CC = E_Q\left[e^{-rt}(S_t^* - K)^+\right] = E_Q\left[E_Q\left[e^{-rt}(S_t^* - K)^+ \,\middle|\, \{T_t\}\right]\right] = E_Q C_t = EC_t,$$

where

$$C_t = S_0 \Phi\left[\frac{\log\frac{S_0}{K} + rt + \frac{1}{2}\sigma^2 T_t}{\sigma\sqrt{T_t}}\right] - Ke^{-rt}\Phi\left[\frac{\log\frac{S_0}{K} + rt - \frac{1}{2}\sigma^2 T_t}{\sigma\sqrt{T_t}}\right] \tag{1.2.1}$$

Φ – distribution function of standard normal distribution,
K – price of carrying out of the option,
r - free from risk rate of return,
T_t – stochastic process.

It is supposed that the process $T = \{T_t : t \geq 0\}$ has stationary increments and thick tails. It can be approximated with the processes, which have asymptotic self-similarity [see [1]].

Expected value P_t in relation to measure Q is the same as in relation to the measure induced by the process of the price of the stock $\{S_t\}$.

The price of the option of selling is determined on the basis of parity purchasing – selling:
the price of the option of selling = the price of the option of purchasing +
- the price of the stock + the current price of carrying out.

For the sake of making comparisons, a rearranged standard Black–Scholes formula for evaluating the European option of purchasing looks like this:

$$C_t = S_t \Phi\left[\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{(T - t)}}\right] - Ke^{-r(T-t)}\left[\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{(T - t)}}\right], \tag{1.2.2}$$

where T means the length of period of validity of the option expressed in years.

The price of the European option of selling is described with the following formula:

$$P_t = -S_t \Phi\left[-\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right] + Ke^{-r(T-t)}\Phi\left[-\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right]. \qquad (1.2.3)$$

Comparing formulae (1.2.1) and (1.2.2) and (1.2.3) we can observe that in the standard Black – Scholes formula occur the process of the price of the basic instrument $\{S_t\}$ and the period of time left before expiry of the option (T-t). Whereas in the formula (1.2.1) occurs S_o – the price of the basic instrument at the initial point of time t = 0. It means that in order to do evaluation it is not necessary to know the complete course of the price of the basic instrument. Besides, in the formula (1.2.1) there is a stochastic process T_t that is ascending and has stationary increments and finite – dimensional distribution with so called "thick tails". This process can be approximated with distribution of random variable $t + t^H(T_1 - 1)$, where H is the Hurst exponent [see [2] p. 324] and T_l has approximated distribution $RG\left(\frac{v}{2}, \frac{v-2}{2}\right)$ that is inverted Gamma distribution with parameters $\frac{v}{2}, \frac{v-2}{2}$, the Brownian motion $W(T_l)$ has distribution $((v-2)/v)^{1/2} T(v, \sqrt{v}, 0)$ where $T(v, \sqrt{v}, 0)$ is the t-Student distribution with v degrees of freedom, $0 < H < 1$ and $v > 4$ [see [1]].

The empirical study basis on the formula:

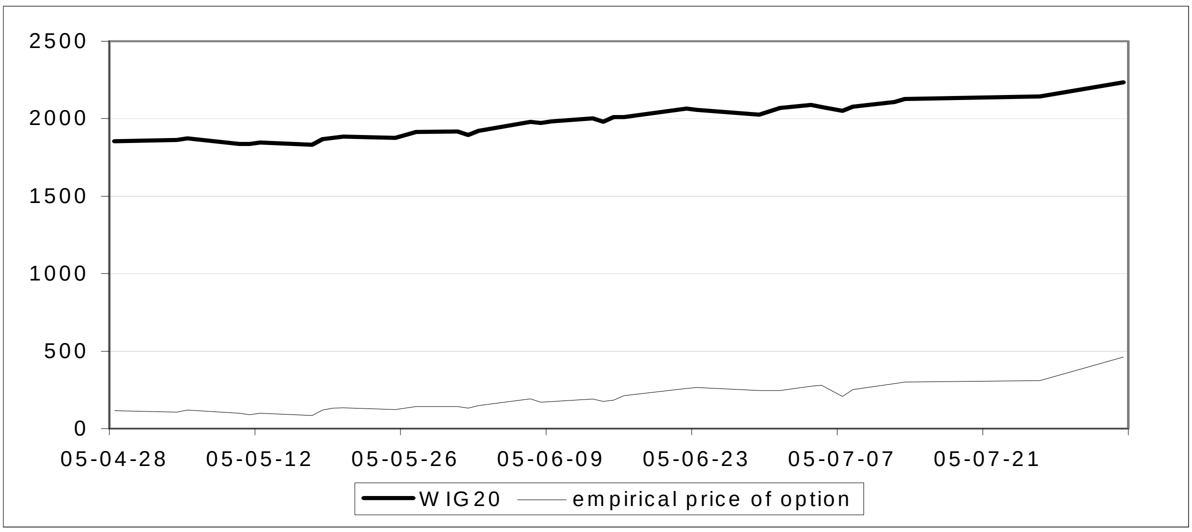
$$C_t = S_t \Phi\left[\frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_t}{\sigma\sqrt{T_t}} \right] - Ke^{-rT_t}\left[\frac{ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T_t}{\sigma\sqrt{T_t}} \right], \qquad (1.2.4)$$

where the process T_t is the same like process in the Heyde and Leonenko formula.

2 Empirical analysis on the basis of WIG 20

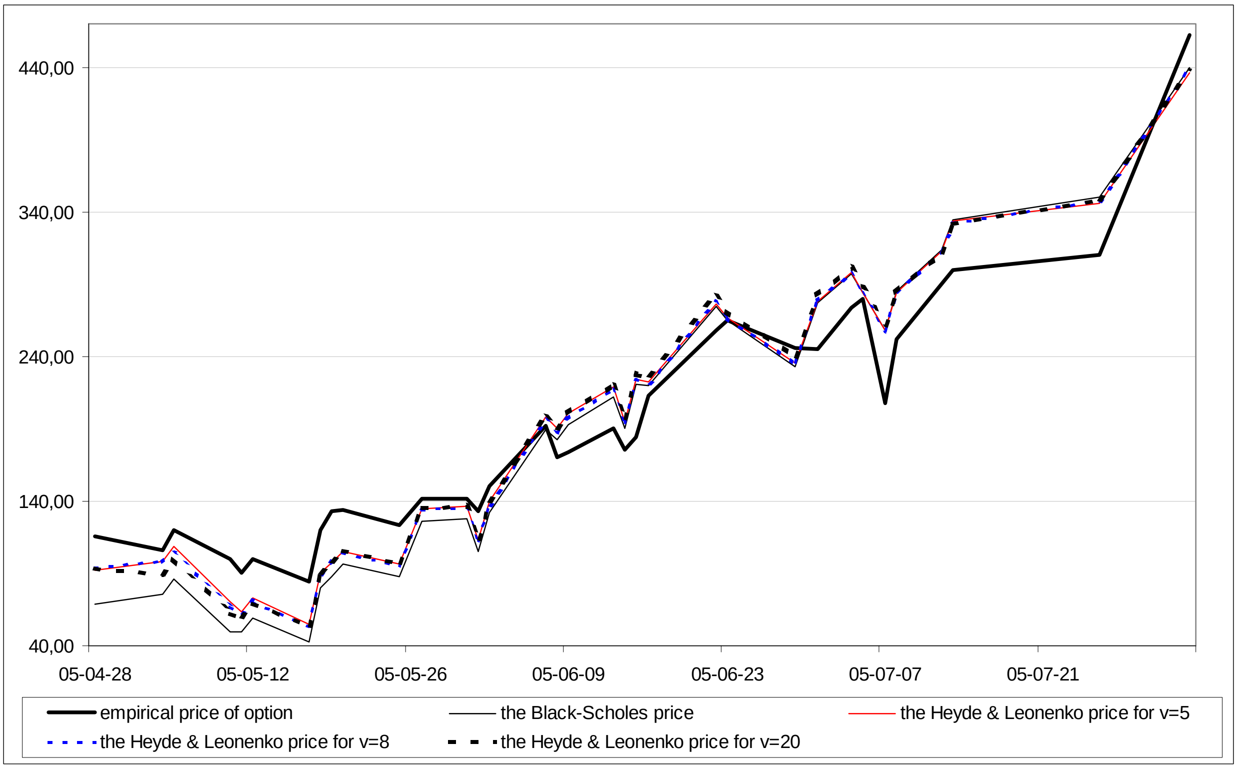
The research was done on the basis of quotations of WIG 20 index (index of Warsaw Stock Market (WGPW)). WIG 20 was introduced in April 1994. It is calculated on the basis of the value of stock portfolio of 20 companies quoted on the primary market. The period o time under research is 28.04.2005 – 03.08.2005. As the derivative was selected the European purchasing option for WIG 20 index (OW20I5180) with the expiry date set for September 2005 and the price of carrying out 1800. In further considerations 34 time series were taken into account. They show the picture of quotations of WIG 20 index and characterise change of the price of the selected purchasing option.

Fig. 2.1 WIG 20 quotations and the prices of the purchasing option – empirical data



In this part of the article there is a comparison of the prices of the European purchasing option of WIG 20 index calculated on the basis of formula (1.2.2) and (1.2.4) with empirical prices taken from stock exchange data as published by WGPW. Besides, there is a description of a way of approximating a stochastic process T_t with generating adequate distributions.

Fig. 2.2 Comparison of the prices of theoretical options with stock exchange data



In figure 2.2 there are theoretical prices of options calculated with the standard Black–Scholes formula and prices calculated on the basis of formula (1.2.4). In an empirical way the parameter v was given the value 5, v = 5. On the basis of an analysis of scaled range it was assumed that the value of the Hurst exponent would be 0,52 [see [2]]. All the calculations approximating the process $T_{t.}$ and evaluating the Hurst exponent were carried out with the spreadsheet software MS Excel.

3 Conclusions

In the process of calculating it was observed that the discussed modification is a better tool for modelling of empirical price of the option of purchasing for WIG 20 index than the standard Black–Scholes formula. It was also concluded that the best matching was possible when v parameter was 5. On the basis of the calculations of the Hurst exponent done for empirical prices of options and taking advantage of examples in literature, it can be observed, that the greater value of the Hurst exponent is the "smoother" time series of the price of exchange securities gets, which means that the risk becomes smaller. Time series of the prices of exchange securities discussed in literature concerning the field of financial engineering usually contain the Hurst exponent of the range $0,5 \le H \le 0,65$.

The calculations and their results give a good basis for stating that formula (1.2.1) is an interesting modification of the standard Black–Scholes model of evaluating options. It reflects real properties of processes of the prices of instruments quoted on stock exchange.

Literature

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Summary

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The popularity of the Black–Scholes model is due to the fact that the prices of derivatives calculated with the formula are usually close to the real prices used on the market. Many modifications of the model enable to give more precise descriptions of newer and more complicated financial instruments. With certainty, in the coming years there will be a great number of works dealing with this issue published by scientists from the fields of economics, financial engineering or econometrics. It leads to the conclusion that the issue of evaluating of financial instruments is still interesting for both representatives of the world, of science as well as for professional participants of capital market.