Some results on pricing of selected exotic options via subordinated Lévy models

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Abstract

Detecting the fair, ie. no-arbitrage, price of an option is a very interesting and challenging task of quantitative finance. It results mostly from the fact that the option payoff is nonlinear and the price can be very sensitive to the changes of underlying factors (especially ATM options). This common feature is further stressed in case of options with some discontinuity in the payoff function. By contrast, options that are illiquid can be very sensitive to the asymmetry of the probability distribution of underlying factors as well as its fat tails. A popular model, how to deal with stylized facts of financial asset returns, such as skewness or kurtosis of the option underlying distribution, is a subordinated Lévy model (VG, NIG). In this paper, we apply these two models to estimate the value of several exotic options written on various FX rates.

Keywords

FX rate, variance gamma model, normal inverse gaussian model, option, exotic option.

JEL Classification: C1, C5

1 Introduction

Options are quite important type of financial derivatives since they allow to fit even very specific fears (hedging) and outlooks (speculation) about the future evolution. Due to the nonlinear payoff function and potential high sensitivity to changes in the input factors, such as volatility or even maturity, options are very challenging also for modeling purposes.

Obviously, since the standard option valuation model of Black and Scholes (and Merton) was based on the assumption of normally distributed returns, the presence of skewness and kurtosis at the market complicates the situation significantly. A common market practice is to use the market price as an exogenous variable to be put into the BS formula (Black and Scholes, 1973). Thus, a so called *implied volatility* is obtained, ie. a number that assures that BS model provides the right price. Such implied volatility can subsequently be used to value even exotic options, which are not traded at the market.

An alternative approach is to utilize some of the models that allow us to take into account also the higher moments of the underlying distribution, such as, at least in Finance, popular examples of Lévy models, the VG model and the NIG model.

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In this paper we focus on valuation of selected types of exotic options assuming three kinds of models, BS, VG, and NIG. In particular, we are interested in the impact of various real combinations of the underlying volatility, skewness and kurtosis on the price of digital and Asian options. In the subsequent options, the option terminology is briefly reviewed. After that, the BS plain vanilla option value model is stated and redefined in order to incorporate also VG and NIG underlying processes. Finally, various FX rates are supposed as an underlying asset of ATM, ITM and OTM options.

2 Option terminology

Options are nonlinear types of financial derivatives, which gives the holder the right (but not the obligation) to buy the underlying asset in the future (at maturity time) at prespecified exercise price. Simultaneously, the writer of the option has to deliver the underlying asset if the holder asks. Option analysis is subject of study of almost any book dealing with financial markets and also in most of the books focusing on quantitative finance. In this section, we mostly follow Tichý (2008, 2011).

Options can be classified due to a whole range of criterions, such as counterparty position (short and long), maturity time, complexity of the payoff function, etc. The basic features are *the underlying asset* ($\mathcal{S}$), which should be specified as precisely as possible (it is important mainly for commodities),[2] *the exercise price* ($\mathcal{K}$), and *the maturity time* (T).

If the option can be exercised only at maturity time T, we call it *the European option.* By contrast, if it can be exercised also at any time prior the maturity day, ie. $t \in [0, T]$, we refer to it as *the American option.* A special type of options, possible to be classified somewhere between European and American options is *the Bermudan option,*/ which can be exercised at final number of times during the option life.

In dependency on the complexity of the payoff function, we usually distinguish simple *plain vanilla options* (PV) and *exotic options.* However, by a plain vanilla option we generally mean call and put options with the most simple payoff function.[3] Thus,

$$\Psi_{call}^{vanilla} = (\mathcal{S}_T - \mathcal{K})^+ \tag{1}$$

for vanilla call, and

$$\Psi_{put}^{vanilla} = (\mathcal{K} - \mathcal{S}_T)^+ \tag{2}$$

for vanilla put, where $(x)^+ \equiv \max(x; 0)$.

Due to the definition of an option – it gives a right, but not an obligation to make a particular trade – we can deduce basic differences between the short and the long position. While the payoff resulting from the long position is non-negative, either 0 or $\mathcal{S}_T - \mathcal{K}$, the payoff of the short position will never be positive, ie. it is either $\mathcal{K} - \mathcal{S}_T$ or 0. Moreover, it is obvious, that the long call payoff is not limited from above, but the short position payoff function goes only up to the exercise price (underlying asset price is zero).

Any financial option with more complex payoff function than is the one of a standard European (American) call or put option is referred to as *the exotic option.* The majority of exotic options is traded outside organized exchanges, at so called OTC markets. However, several types of exotics are so popular that the major derivatives exchanges have listed them (e.g. some options with barriers).

By contrast, many exotic options are so unique that they are suitable only for investors for whom they were originally designed. Thus, within the payoff pattern special needs or believes and fears of corporate or institutional investors are respected. This fact decreases the liquidity significantly and also the pricing and hedging procedure can be substantially complicated.

[2]If we do not state otherwise, it will be supposed that the underlying asset is a non-dividend stock.
[3]Sometimes, by plain vanilla options we mean any option which is regularly traded at the market.

Among exotic options we can assign:

- *Package* – the portfolio, which consists of calls, puts, forwards, cash or underlyings;
- *Multistage options* – allow the holder to make some decision in the future, such as *compound option*, ie. the option on an option, *chooser option* that gives the chance to choose between a call and a put option, etc.;
- *Digital* (*binary*) *options* – the payoff is either *everything* or *nothing*;
- *PD options* (*path dependent options*) – the payoff depends on the path followed by the underlying asset price during the option life;
- *and many others.*

Path dependent options. This miscellaneous family of exotic options consists of contracts whose payoff depend not only on the price at the maturity time but also on the past – the path which was followed by the underlying asset price during the life of the option. The most popular examples are options:

- paying off if a prespecified barrier level has (not) been reached by the underlying asset price (*Barrier options*),
- with the payoff determined on the basis of the most favourable underlying asset price (*Lookback options*),
- with the payoff on the basis of the average underlying asset price (*Asian options*).

A specific problem that arises in pricing of PD options is that the standard pricing procedures are based on continuous monitoring of the underlying asset price. By contrast, the payoff is usually specified in such a way, that the path consists of daily (weekly) closing prices.

Digital options. Although digital options are not strictly path dependent, they are closely related to barrier options in some way. The holder of such option will get payoff if the underlying asset price $\mathcal{S}_T$ will reach or exceed the exercise price $\mathcal{K}$ at maturity (simple digital options) or, alternatively, at any time prior the maturity (barrier digital options). The payoff can be prespecified amount $\mathcal{Q}$, selected asset (usually the underlying one), or it can be specified in another way.

The most important feature is that the payoff does not depend on the amount by which the underlying asset price $\mathcal{S}_T$ exceeds the exercise level $\mathcal{K}$. The payoff is either everything or nothing.

The most simple case of digital options is *a cash-or-nothing call.* The holder of such option receives the amount $\mathcal{Q}$ if the underlying asset price at maturity time T is higher than $\mathcal{K}$. The payoff $\Psi_{call}^{dig/cash}$ can be formulated as follows:

$$\Psi_{call}^{dig/cash}(\mathcal{S},\mathcal{K};\mathcal{Q}) = \begin{cases} \mathcal{Q} & \text{if} \quad \mathcal{S}_T \geq K \\ 0 & \text{if} \quad \mathcal{S}_T < \mathcal{K}. \end{cases}$$

Another case is *an asset-or-nothing call*, without fixed payoff amount in advance – it is (the value of) the underlying asset price at maturity. In Table 1 we review the basic pay-offs of digital options. More specific type of digital options is *a gap option.* It is an option with a gap in the payoff.

Analyzing the relation between $\Psi_{call}^{dig/asset}$ and $\Psi_{call}^{dig/cash}$, we can uncover the following parity:

$$\mathcal{V}_{call}^{dig/asset}(\tau;\mathcal{S},\mathcal{K};\mathcal{S}) = \frac{\mathcal{K}}{\mathcal{Q}}\mathcal{V}_{call}^{dig/cash}(\tau;\mathcal{S},\mathcal{K}) + \mathcal{V}_{call}^{vanilla}(\tau;\mathcal{S},\mathcal{K}). \tag{3}$$

Similar parity condition can be identified also for put options.

Table 1: Payoff function of basic types of digital options

Option	Cash-or-nothing		Asset-or-nothing	
Relation of $\mathcal{S}_T$ and $\mathcal{K}$	call	put	call	put
$\mathcal{S}_T > \mathcal{K}$	$\mathcal{Q}$	0	$\mathcal{S}_T$	0
$\mathcal{S}_T = \mathcal{K}$	$\mathcal{Q}$	0	$\mathcal{S}_T$	0
$\mathcal{S}_T < \mathcal{K}$	0	$\mathcal{Q}$	0	$\mathcal{S}_T$

Asian options. An Asian option[4] is another example of *strongly path dependent* options. Also within this group of exotics, various types can be distinguished.

From one of the points of view, we distinguish which part of the payoff function is given by the average of monitored underlying prices, $\mathcal{S}_{Ave}$, either the exercise price or the quantity, from which we deduce the fixed exercise price.

Define a binary variable η indicating whether the option is call ($\eta = 1$) or put ($\eta = 1$). Then, for the continuously calculated average over the period τ we can formulate the Asian option payoff (with exercise price $\mathcal{K}$) as follows:

$$\Psi^{As,fix,con}_{call/put} = \left[\eta\left(\frac{1}{T}\int_0^T \mathcal{S}_t \mathrm{d}t - \mathcal{K}\right)\right]^+ . \tag{4}$$

Similarly, for the case of discretely monitored Asian option, ie. $t = \{t_0, t_1, \ldots, T\}$, we get:

$$\Psi^{As,fix,dis}_{call/put} = \left[\eta\left(\frac{1}{n}\sum_{i=0}^{n} \mathcal{S}_{t_i} - \mathcal{K}\right)\right]^+ . \tag{5}$$

By contrast, having a floating exercise price on the basis of discretely monitored underlying asset prices, we get:

$$\Psi^{As,flo,dis}_{call/put} = \left[\eta\left(\mathcal{S}_T - \frac{1}{n}\sum_{i=0}^{n} \mathcal{S}_{t_i}\right)\right]^+ . \tag{6}$$

When we specify the option features, the way in which the average is determined, should be of particular interest. We should distinguish among arithmetical and geometrical averages. For example, assuming discrete monitoring, we get:

$$\mathcal{S}^{art}_{ave} = \frac{1}{N}\sum_{n} \mathcal{S}_n,$$

and

$$\mathcal{S}^{geo}_{ave} = \sqrt[n]{\prod \mathcal{S}_n},$$

respectively. Alternatively, it can be specified that the weights of particular asset prices are not equal, but time dependent – so that we get weighted average.

Asian options are used mainly in connection with continuous (regular) consumption of commodities (the risk of average price) and at markets with low liquidity (to reduce the risk of large trade impacts).

3 Analytical valuation

Although the option valuation formula can be derived by various approaches, such as utilization of risk neutral expectations or solving of partial differential equations, it must – under the same conditions – always lead to the same result. In this section, valuation formulas assuming three common kinds of underlying distribution will be provided.

[4]The term *average option* can be alternatively used. Asian options were originally introduced in 1987 by *Banker's Trust Tokyo* on a crude oil.

BS model. Assuming the payoff function of plain vanilla call:

$$f_T = \Psi^{vanilla}_{call} = (\mathcal{S}_T - \mathcal{K})^+ \,,$$

we get the valuation formula as follows (*BS model for vanilla call*):[5]

$$\mathcal{V}^{vanilla}_{call}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) = \mathcal{S}\mathcal{N}(d_+) - e^{-r\tau}\mathcal{K}\mathcal{N}(d_-). \tag{7}$$

Similarly, assuming the payoff function of vanilla put, we get (*BS model for vanilla put*):

$$\mathcal{V}^{vanilla}_{put}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) = e^{-r\tau}\mathcal{K}\mathcal{N}(-d_-) - \mathcal{S}\mathcal{N}(-d_+). \tag{8}$$

In both cases:

$$d_\pm = \frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{9}$$

Moreover, $\mathcal{S}$ is the underlying asset price at the valuation time (t) and it is supposed to follow log-normal distribution, τ is the time to maturity (ie. $\tau = T - t$), r is riskless rate valid over τ, σ is the volatility expected over the same period, both *per annum*, and $\mathcal{N}(x)$ is distribution function for standard normal distribution

VG model. While BS model is based on the geometric Brownian motion, and thus the unrealistic assumptions of Gaussian distribution, more advanced VG model[6] allows us to fit also the skewness and excess kurtosis of the returns. Recall from Tichý (2010, 2011) the VG process $\mathcal{VG}(g(t;\nu); \theta, \vartheta)$ as follows:

$$\mathcal{VG}_t = \theta g_t + \vartheta Z(g_t) = \theta g_t + \vartheta\sqrt{g_t}\varepsilon. \tag{10}$$

However, for pricing issue the risk neutral probability density is assumed and VG process must be compensated to get risk neutral drift:[7]

$$\mathcal{S}^{(\mathcal{Q})}_T = \mathcal{S}_t \exp\left(rt + \theta g_t + \vartheta\sqrt{g_t}\vartheta - \varpi t\right),$$

where $\varpi = -\frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\vartheta^2\nu\right)$.

Since the VG process can be obtained as subordinated Brownian motion, the option valuation formula can be written in quite easy way. Recall first the density function for gamma distribution $\mathcal{G}(\mu, \nu)$ with $\mu = 1$,

$$f_{\mathcal{G}}(g) = \frac{g^{\frac{t}{\nu}-1}\exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})},$$

which can be used in the BS formula above if we redefine it as a function of underlying asset price, volatility and time to maturity, $\mathcal{V}^{\mathcal{BS}}(\tau; \mathcal{S}, \sigma)$. And thus, the VG option pricing model, $\mathcal{V}^{\mathcal{VG}}(\tau; \mathcal{S}, \vartheta)$, can be formulated as:

$$\mathcal{V}^{\mathcal{VG}}(\tau; \mathcal{S}, \vartheta) = \int_0^\infty f_{\mathcal{G}}(g)\mathcal{V}^{\mathcal{BS}}\left(\tau; \mathcal{S}\exp\left(\theta g + \frac{1}{2}\vartheta^2 g - \omega\tau\right), \vartheta\sqrt{\frac{g}{\tau}}\right) \,\mathrm{d}g \tag{11}$$

Hence, we can interpret the formula as BS model integrated over gamma time.

[5] Black and Scholes (1973); an alternative model for dividend paying stocks is due to Geske (1978), whereas the case of FX rates was analyzed by Garman and Kohlhagen (1983).

[6] For more details of VG model see eg. Madan and Seneta (1990) for the symmetric case and Madan and Milne (1991) and Madan et al. (1998) for the asymmetry case.

[7] Several approaches of compensators for Lévy model were analyzed by Fujiwara and Miyahara (2003).

NIG model. Since the NIG model[8] can be defined in a similar way to VG model, we can adopt the approach described above. Thus, following again the formulation of Tichý (2011), the option valuation formula for the NIG model case is:

$$\mathcal{V}^{\mathcal{NIG}}(\tau; \mathcal{S}, \vartheta) = \int_0^\infty f_{\mathcal{IG}}(\mathcal{I}) \mathcal{V}^{\mathcal{BS}} \left(\tau; \mathcal{S} \exp\left(\theta \mathcal{I} + \frac{1}{2} \vartheta^2 \mathcal{I} - \omega \tau \right), \vartheta \sqrt{\frac{\mathcal{I}}{\tau}} \right) \mathrm{d}\mathcal{I}. \qquad (12)$$

4 Exotic option pricing

Many users of financial derivatives (hedgers, speculators) can have very special believes or fears about the future evolution. In such cases, exotic options might be more proper to be used within hedging (or speculation) than plain vanilla options. In this section, we provide some results on pricing of digital options and Asian options. While the price of the former is related to the probability of exercising (of almost any kind of) the option, the latter is related to the average price, which can be observed over given time.

The input data are derived from daily observations of FX rate log-returns of CZK (Czech koruna), GBP (British pound), HUF (Hungarian forint), IDR (Indonesian rupiah), JPY (Japanese yen), KRW (Korean won), PLN (Polish złoty) and USD (US dollar) from the point of view of EUR. The time series covers preceding 10 years (2001 – 2010) and the basic descriptive statistics are apparent from Table 2, p.a. if relevant.

Table 2: Basic descriptive statistics of selected daily FX rate log-returns

FX rate	*Mean*	*Median*	*St.dev.*	*Skewness*	*Kurtosis*
CZK	0.0332	0.0392	0.0668	0.0314	10.2736
GBP	−0.0320	−0.0045	0.0809	−0.4706	8.0269
HUF	−0.0047	0.0370	0.1005	−0.6868	14.0017
IDR	−0.0296	−0.0211	0.1396	0.3571	16.0597
JPY	−0.0016	−0.0955	0.1241	0.1799	8.9864
KRW	−0.0312	−0.0114	0.1341	−0.0726	13.8782
PLN	−0.0031	0.0666	0.1108	−0.3377	14.5372
USD	−0.0359	−0.0569	0.1044	0.0331	5.8854

In order to simplify the estimation of the model parameters, options on forward FX rates are assumed which is equivalent to price options on current (spot) FX rate assuming zero risk-less rates in both currencies, or, at least their equality. Moreover, we assume the identity of real and risk-neutral parameters of the higher moments.

We proceed as follows: first, we utilize the standard deviation and potentially also the skewness and kurtosis as described in Table 2; second, we apply GBM, VG and NIG models to price a 4-week ATM call option either via MC simulation (see Tichý (2010) for suggestions of implementation). Moreover, we normalize the starting value of the FX rate to be 1.0.

First, we evaluate digital options (Table 3), ie. we consider the case of option that provides payoff of 1 currency unit (EUR) if the future price of considered FX rate is above the exercise price assuming ATM, OTM and ITM options as above. This time, however, we can observe more or less (positive) difference in the value of the option (or probability of exercising the vanilla call) in case of ATM's and ITM's, while considering OTM's VG or NIG models provide us often lower values. Note however, that we can interpret it as the implication of the shape of the probability distribution of given FX rate (what is the probability of more/less far values to the mean/exercise price).

[8]More details on the NIG model can be found in Barndorff-Nielsen (1995), (1998), etc.

Table 3: Values of selected digital call options

FX rate	*CZK*	*GBP*	*HUF*	*IDR*	*JPY*	*KRW*	*PLN*	*USD*
Model	ATM option							
BS	0.497	0.496	0.496	0.491	0.493	0.493	0.495	0.493
VG	0.521	0.575	0.572	0.471	0.372	0.559	0.568	0.478
NIG	0.511	0.542	0.543	0.485	0.420	0.532	0.556	0.482
Model	5% OTM option							
BS	0.998	0.991	0.982	0.900	0.933	0.945	0.962	0.951
VG	0.991	0.973	0.964	0.922	0.965	0.939	0.944	0.956
NIG	0.992	0.976	0.967	0.924	0.967	0.942	0.946	0.957
Model	5% ITM option							
BS	0.003	0.012	0.022	0.105	0.072	0.060	0.044	0.054
VG	0.007	0.011	0.016	0.081	0.074	0.042	0.018	0.057
NIG	0.006	0.009	0.014	0.079	0.070	0.042	0.016	0.055

The next example is focused on Asian options. In Table 4 we provide estimated prices assuming ATM/ITM/OTM call options with fixed exercise price, ie. the payoff is given as follows (discretely determined on 1-week basis assuming 4 weeks call option):

$$\Psi_{call}^{As,fix,dis} = \left[\eta\left(\frac{1}{n}\sum_{i=0}^{n}\mathcal{S}_{t_i} - \mathcal{K}\right)\right]^{+}.$$

Due to the table, it is apparent that the estimated prices are (almost) the same for models we consider (assuming 3 decimal points), though, assuming six places, we can obtain results leading to very similar conclusions as above. Generally, the reason here is the main feature of Asian options – it is related to the average of the prices, and thus it should be less sensitive to fluctuation as well as the selection of the proper model.

Table 4: Values of selected types of Asian call options

FX rate	*CZK*	*GBP*	*HUF*	*IDR*	*JPY*	*KRW*	*PLN*	*USD*
Model	ATM option							
BS	0.002	0.003	0.004	0.006	0.007	0.007	0.008	0.009
VG	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.008
NIG	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.008
Model	5% ITM option							
BS	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
VG	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
NIG	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050
Model	5% OTM option							
BS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
VG	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
NIG	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Model	Floating exercise option							
BS	0.003	0.005	0.007	0.010	0.011	0.013	0.014	0.015
VG	0.003	0.005	0.007	0.009	0.011	0.013	0.014	0.015
NIG	0.003	0.005	0.007	0.009	0.011	0.013	0.014	0.015

5 Conclusions

The presence of jumps and unexpected decreases (increases) in price provide very challenging task on any risk and pricing model. An interesting approach to evaluate the real (and risk-neutral) aspects of the returns of the underlying asset of an option more properly is to assume some of the subordinated Lévy models.

In this paper the impact of various combinations of volatility/skewness/kurtosis was evaluated in order to estimate the value of several exotic options. The results are an important contribution to efficient pricing and hedging of exotic options, especially when compared to too simplifying implied volatility approach.

References

[1] BARNDORFF-NIELSEN, O. E. (1995). Normal inverse Gausian distributions and the modeling of stock returns, *Research report* No. 300. Aarhus: Department of Theoretical Statistics, Aarhus University.

[2] BARNDORFF-NIELSEN, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**: 41–68.

[3] BLACK, F., SCHOLES, M. (1973). The pricing of options and corporate liabilities, *Journal of Political Economy* **81**: 637–659.

[4] FUJIWARA, T., MIYAHARA, Y. (2003). The minimal entropy martingale measures for geometric Lévy processes. *Finance and Stochastics* **7**: 509–531.

[5] GARMAN, M., KOHLHAGEN, S. (1983). Foreign currency option values. *Journal of International Money and Finance* **2**: 231—237.

[6] GESKE, R. (1978). The pricing of options with stochastic dividend yield. *Journal of Finance* **25**: 383–417.

[7] MADAN, D. B., MILNE, F. (1991). Option pricing with VG martingale components. *Mathematical Finance* **1**: 39–56.

[8] MADAN, D. B., SENETA, E. (1990). The VG model for share market returns. *Journal of Business* **63**(4): 511–524.

[9] MADAN, D. B., CARR, P., CHANG, E. C. (1998). The variance gamma process and option pricing. *European Finance Review* **2**: 79–105.

[10] TICHÝ, T. (2008). *Lattice models – Pricing and Hedging at (In)complete Markets.* Ostrava: VŠB-TU Ostrava.

[11] TICHÝ, T. (2011). *Lévy models – Selected applications with theoretical background.* SAEI Vol. 9 (Series on Advanced Economic Issues). Ostrava: VŠB-TU Ostrava.

[12] TICHÝ, T. (2010). *Simulace Monte Carlo ve Financích: Aplikace při ocenění jednoduchých opcí.* SAEI Vol. 6 (Series on Advanced Economic Issues). Ostrava: VŠB-TU Ostrava.