The Error Modelling for the Forecasting of the Mortality Index

Modelování chyby při předpovědi indexu úmrtnosti

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Abstract

The article deals with life underwriting risk management. Forecasting mortality is an important issue for insurance companies and pension funds to manage their risks following from contracts between policyholders and companies. For the insurance companies the new solvency rules will be implemented in 2013. Within the Solvency II the insurance companies should determine the capital requirement for mortality/longevity risks.. The insurance companies should use a stochastic approach for an assessment of mortality/longevity risks. The Lee-Carter model (1992) is still used and popular approach for forecasting mortality rate. The aim of the paper is to apply the Lee-Carter model for the mortality tables for males and females in the Czech Republic and to model the error for forecasting of mortality index. The error terms of the model for forecasting mortality are assumed to follow Gaussian distribution. But the error terms are not Gaussian distributed on the basis of characteristics such as skewness or kurtosis.

Key words

Mortality risk, longevity risk, Solvency II, Lee-Carter model, VG model.

JEL Classification: G22

1. Introduction

The evaluation of mortality rate has a significant impact on the amount of the reserves resulting from a contract between a policyholder and an insurance company or a pension fund. In the last decades we can observe a decrease in probability of death at old ages. For appropriate determining of expected present cash flow, it is required a suitable model for mortality forecasting to avoid an underestimation of future cost. We should devote to the model that allow capturing the systematic deviation from given trends in the mortality rate.

The articles relating to the forecasting mortality rate using stochastic models have been published since 90´s of the last century. The most well-know and still used approach is Lee-Carter model.

The Lee-Carter model (Lee and Carter, 1992) is very useful and still used approach for mortality risk modeling. The Lee-Carter model was originally applied for US mortality data for period 1933 – 1987. This model was successfully used for a long-time forecasting of mortality rates in more countries for different time period, for example Canada (Lee and Nault, 1993), Chile (Lee and Rofman, 1994) or Japan (Wilmoth, 1996).

The aim of the paper is to apply the Lee-Carter model for the mortality tables for males and females in the Czech Republic and to model the error for forecasting of mortality index.

The paper is structured as follows. In the first chapter we devote to the impact of life insurance risk on the required capital of an insurance company. The second chapter is devoted to the characterization of the Lee-Carter model for the long-run forecasting. In the next

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chapter we will apply the model for the Czech data and we make analysis of the error for mortality forecasting.

2. Life insurance risk and its impact on the insurance capital

The new European solvency regulation, Solvency II, changes an assessment of capital requirement. According to this directive an insurance company should take into account all risks that insurer can be exposed to. The key role of an insurer is the promise of the financial compensation at some moment in future. Hence an insurance company need adequate amount of capital. Capital needs we can distinguish from a number points of view:

- regulatory: which follows from directive;
- rating agency: which is related to the probability of insolvency and the ability to continue with the current rating;
- going concern: which is related to the ability to continue with the future business plan.

After implementing the Solvency II, the required capital corresponds to the Value at Risk on the probability level 99.5 % for one year horizon. The rating agencies focus on the Expected Shortfall for the lower probability level.

There were stated the several reasons why the insurance companies need to hold the capital. Firstly, the capital serves to absorb the extreme unexpected losses. Secondly, the supervisory body requires capital which serves to protect the policyholder. Considering a life insurance risk, which is the risk resulting from the liabilities sites. Within the life insurance, the technical provisions are created in the moment of the concluding a contract between an insurer and a policyholder.

The life underwriting risk can be defined as a decrease in value due to different mortality than expected or due to a change in the mortality expectation. The traditional products of the life insurance are term assurance, life annuities and so on.

The life underwriting risk is related to death or longevity of a person, so the important risks to be measured are mortality and longevity risk. The mortality risk comes to question in case of the sooner death of the insured than the insurance companies expected. The longevity risk arises when the insured lives longer than expected due to, which more benefit have to be paid. The importance of the longevity risk is pointed out in the Solvency II directive. Mortality tables are the most suitable instrument for measuring life expectancy.

3. The Lee-Carter model

The Lee-Carter model [9] is defined as the logarithm of the central death rate at the age *x* and for a given year *t* as follows

$$\ln(m_{xt}) = a_x + b_x k_t + \varepsilon_{xt}, \quad t = 1,2,...T, \qquad x = 1,2,..n. \tag{1}$$

where a_x, b_x, k_t are the parameters and ε_{xt} is error term assuming Gaussian distribution $N(0,\sigma^2)$. The random term ε_{xt} reflects age-specific influence not captured by the model. The parameters a_x are age specific constants, which represent general age shape of mortality. This parameter is determined as an arithmetic average logarithms of the central death rate at the age *x* and for a given year *t* for whole period as follows

$$a_x = \frac{1}{T}\sum_{t=1}^{T}\ln(m_{xt}) \tag{2}$$

So, for a unique solution of the model, we should take into account these conditions: the sum of the square values of the b_x is equal to unique, $\sum_x b_x^2 = 1$ and sum of k_t values is equal to zero, $\sum_t k_t = 0$.

The parameter k_t indicates the time trend in the general mortality level. The parameter b_x reflects the changes of the mortality at the age x in response to how the mortality index varies. For the estimation of the parameters b_x, k_t the Singular Value Decomposition (SVD) is originally used. Naturally, the parameters can be estimated using by Maximum likelihood or Weighted least square method. These mentioned methods are the most used for estimating the parameters of the model. However, Koissi et al. [8] have compared all these methods and the results for the data of the Nordic countries have shown that SVD is the best alternative for the estimation of the mortality index, k_t. So, for our purpose, we used the SVD method for matrix

$$Z_{x,t} = \ln(m_{x,t}) - \hat{a}_x = b_x k_t . \quad (3)$$

By an application of the SVD method for matrix $Z_{x,t}$ we obtain three matrixes

$$SVD(Z_{x,t}) = ULV' = L_1 U_{x1} V_{t1} + \ldots + L_X U_{xX} V_{xT} \quad (4)$$

where matrix U represents age component matrix, L is matrix of the singular values and matrix V refers to time component. The parameter $\hat{b}_x$ is derived from the first vector of the matrix regarding the age component, $\hat{b}_x = U_{x1}$ and the parameter $\hat{k}_t$ is set as the first vector product of the time component matrix and the first singular value.

For mortality forecast, we assume that $\hat{b}_x$ remains constant. Lee and Carter use forecasts of mortality index, $\hat{k}_t$ from a standard univariate time series model, ARIMA model (0,1,0). They found that a random walk with drift described mortality index very well. The equation is as follows:

$$\hat{k}_t = \hat{k}_{t-1} + \theta + \xi_t , \quad (5)$$

where θ is drift parameter and its stated follows $\theta = (\hat{k}_T - \hat{k}_1)/T - 1$ and for $\xi_t \approx N(0, \sigma_{rw}^2)$ is supposed Gaussian distribution with mean 0 and variance $\sigma_{rw}^2 = \frac{1}{T-1} \sum_{t=1}^{T-1} (\hat{k}_{t+1} - \hat{k}_t - \hat{\theta})^2$

Forecast mortality index $\hat{k}_{T+(\Delta t)}$ at time $T + (\Delta t)$ we use the equation

$$\hat{k}_{T+(\Delta t)} = \hat{k}_T + (\Delta t)\hat{\theta} + \sqrt{\Delta t}\, \xi_t . \quad (7)$$

For forecast of future lognormal central death rate it is possible to use

$$\ln(m_{x,T+\Delta t}) \approx \hat{a}_x + \hat{b}_x \hat{k}_{T+(\Delta t)} . \quad (8)$$

4. Application part

In this part we will devote to forecasting mortality using the Lee-Carter model. Then, we make an analysis of error for forecasting modelling.

4.1 The data

Mortality tables for male and female are obtained from the Czech statistical office are available for the period 1920 – 2009. However, the period 1938 – 1945 is missing due to the World War II.

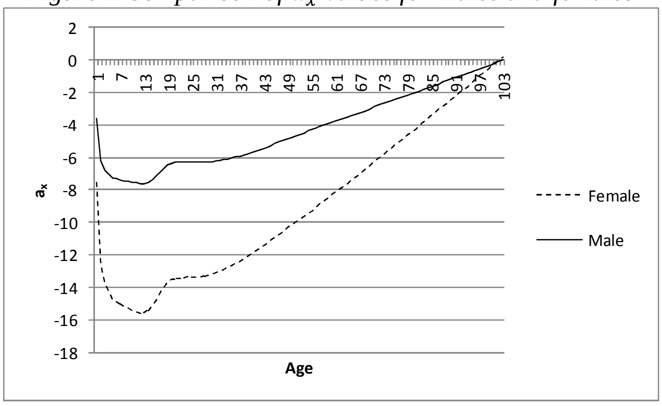
4.2 Parameters Estimation

Firstly, the Lee-Carter model will by applied for the parameters estimation for period 1920 – 2009 for the age between 0 – 102. In order to be made the parameters estimation, it is necessary from available data to determine central death rate $m_{x,t}$ at the age x and for a given year t using the equation for computation of death probability,

$$q_{x,t} = 1 - e^{-m_{xt}} \tag{9}$$

After that, we can determine parameter $\hat{a}_x$ according to (2). The evaluation of this parameter for both males and females is presented in figure 1.

Figure 1 Comparison of a_x values for males and females



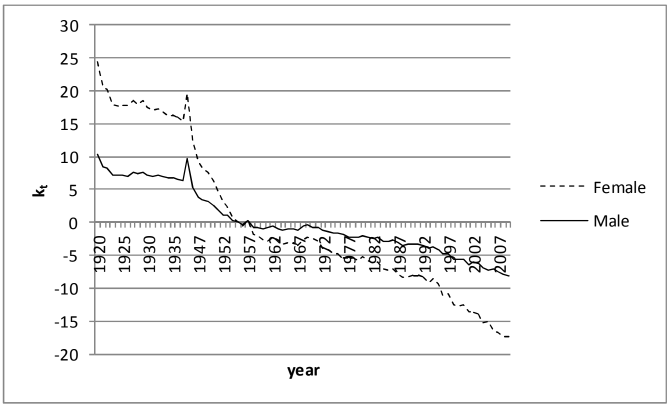
The estimated parameters $\hat{a}_x$ are age specific components that describe general age shape of mortality. From this figure, we can see, that both males and females have general upward trend of mortality, while the younger ages have lower mortality rate and the older ages have higher mortality rate. For the youngest ages and the oldest ages are parameters values identical. For middle ages the parameter is higher for males than females.

Now, we can estimate parameters $\hat{b}_x$ and $\hat{k}_t$, we firstly have to determine matrix $Z_{x,t}$, see (3). This matrix will be decomposed to the three matrixes, see (4). In the following figures (2 and 3) are plotted firstly evaluation of parameters $\hat{b}_x$ and subsequently $\hat{k}_t$.

Figure 2 Comparison of the b_x values for males and females

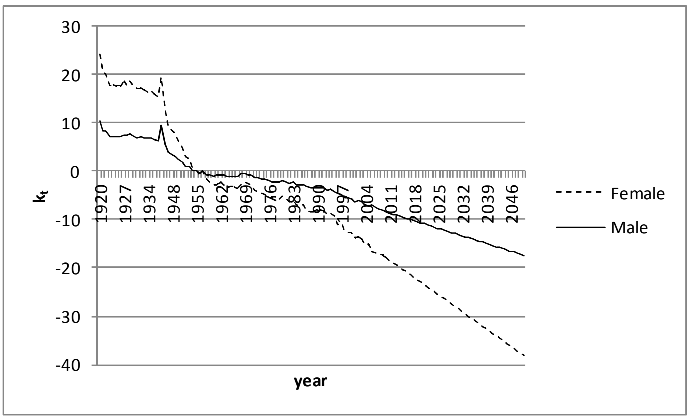
The parameters $\hat{b}_x$ are an age-specific components that represent how rapidly or slowly mortality at the age x varies, when the mortality index, $\hat{k}_t$ changes. Beta is the largest generally for an infant mortality and for males than females. The mortality index, $\hat{k}_t$ expresses the main time trend. The decreasing trend shows an improvement of mortality during observed period. If beta is large for some age, then the death rate at this age varies significantly in response to the changes of the general level of mortality.

Figure 3 Comparison of the k_t values for males and females



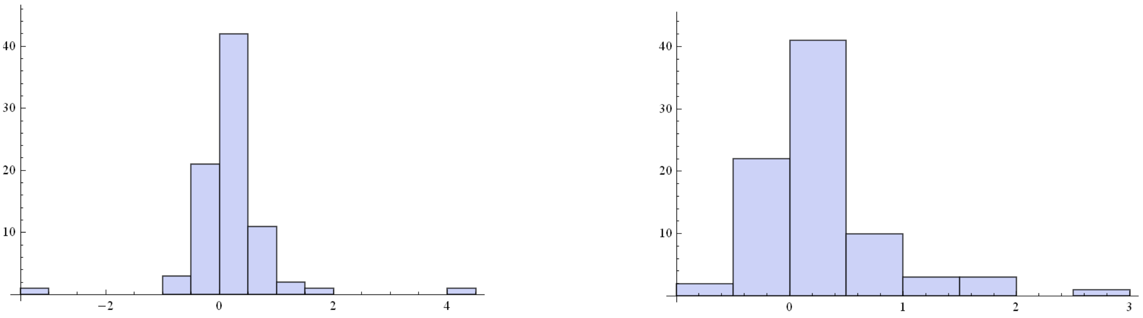
The forecasting of mortality index is made by 2050 according to equation (5). The evaluation of mortality index for males and females is shown in the following figure.

Figure 4 The forecasting of mortality index



In the following figure, there are represented the histograms of $\Delta k_t = k_{t+y} - k_t$ and we can see that parameter Δk_t does not follow Gaussian distribution.

Figure 5 Comparison of the Δk_t *for males and females*



4.3 Analysis of error for mortality modeling

The importance of this analysis is that on the basis of determined distribution characteristics to indicate the suitable model for risk modelling. For analysis of we assume the following equation

$$\xi_t = \hat{k}_t - \hat{k}_{t-1} - \theta, \qquad (10)$$

where $\xi_t \in N\left(0, \sigma^2\right)$.

The basic characteristics – the variance, the skewness and the kurtosis are determined for an error for forecasting mortality, see table 1. The kurtosis for males is significantly higher than for females, because of the change in the mortality after the war.

Table 1: The basic characteristics

	Males	Females
Variance	0.71713	0.56129
Skewness	1.115592	1.61912
Kurtosis	19.17076	5.06183

From the stated results, it is apparent, that errors are not Gaussian distributed. The models that allow us to model the higher moments of the probability distribution, such as skewness and kurtosis, are Levy models. For error term modelling we can use the equation for Variance Gamma model

$$\xi_t = \theta g_{\mathrm{T}} + \vartheta \sqrt{g_{\mathrm{T}}}\, z - \theta_T, \qquad (11)$$

where z is Gaussian distributed, $z \in N(0,1)$. The parameters for VG are estimated by using the generalized method of moments, see table 2.

Table 2: The estimated parameters for VG model

	Males	Females
ϑ	0.7071817	-0.0000097
θ	0.0526457	0.93192207
υ	5.112417	0.362757

Positive skewness means that distribution has a longer tail to the right than to the left. The kurtosis, fourth movement, measures fat tail behavior. The higher is kurtosis, the fatter are tails. The parameter υ allows to drive kurtosis and parameters θ controls skewness.

5. Conclusion

The aim of the paper was to apply the Lee-Carter model for the mortality tables for males and females in the Czech Republic and to model the error for forecasting of mortality index.

Although the changes in mortality rates are relatively small, especially in the developed countries, the mortality rates are relatively stable, which can have a significant influence on the compensation for the clients. For the institutions such as the insurance companies or the pension funds the estimation and forecast mortality rate is important part of longevity/mortality risk management. We detect a suitable model, i.e. Variance Gamma model for modeling an error term for forecasting mortality rate The Lee-Carter model is one of the most used approaches for mortality rate modelling.

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Summary

Příspěvek se zabývá řízením životního upisovacího rizika. V rámci směrnice Solvency II by pojišťovny měli stanovovat kapitálový požadavek pro riziko úmrtnosti a dlouhověkosti.

Pro stanovení těchto rizik lze použít stochastický přístup. Jedním ze stále používaných modelů pro předpověď míry úmrtnosti je Lee-Carterův model. Cílem příspěvku bylo aplikovat Lee-Carterův model na základě úmrtnostních tabulek pro muže a ženy v České republice a modelovat chybu pro předpověď indexu úmrtnosti.