Dependence of the changes in some measures of the variability and location of insurance payments of a cedant on a change of priority in non-proportional reinsurance

Závislosť zmien niektorých mier variability a polohy poistných plnení cedanta od zmeny priority pri neproporcionálnom zaistení

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Abstract

The contribution deals with a study of dependence changes of the mean, variance and coefficient of variation of the insurance payments of a cedant on a change of priority in WXL/R reinsurance (Working Excess of Loss Cover per Risk) with or without an upper limit of the insurance payment of the reinsurer. In the second case, in addition, it analyses the dependence of a change in the upper limit of the insurance payment of the reinsurer in relation to a change of priority with of the constraint of an unaltered expected cedant insurance payment.

Key words

non-proportional reinsurance, WXL/R reinsurance, priority, mean, variance, coefficient of variation

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1 Introduction

Reinsurance is, broadly speaking, the insurance of insurance companies [2]. The traditional reason for reinsurance is related to the diversification of a part of an unacceptably high risk of the insurer, also called a cedant, to the reinsurer. Reinsurance has more functions. In addition to protecting the cedant from a catastrophic loss or cumulative losses, it enables the cedant to achieve a greater homogeneity of the insurance portfolio, thereby reducing the adverse effect of fluctuations in the claim performance (stabilisation role), it enables to increase the volume of a cedant's business, to allow the insurer to market new insurance products, it allows an easy start in business for new insurance companies in the insurance market, it enables the cedant to avoid the creation of excessive insurance reserves, and it enables methodical guidance of the insurer by the reinsurer, if necessary.

Besides the traditional reasons, reinsurance may be used also for other reasons, which are related, for example, to tax reduction and circumventing Insurance Regulatory requirements, and others.

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2 Analysis of excess of loss reinsurance of individual risks without an upper limit on reinsurer payments

Excess of loss reinsurance without upper limit is a form of reinsurance whereby the reinsurer indemnifies the cedant for the amount of a loss above a stated excess point, usually called a priority. The probability that the amount of a loss is greater than the priority is usually small. The reinsurance premium is usually determined according to this probability.

The mentioned insurance protects a cedant from individual losses exceeding this priority. It is convenient for insurance portfolios which are seldom endangered by large losses.

In the case that a claim arises with insurance payment *X*, then the insurance payment for the cedant X_C and reinsurer X_Z is given by the formulae

$$X_C = \min\{X, \alpha\}, \quad X_Z = \max\{X - \alpha, 0\}, \tag{1}$$

where α is the priority of the cedant.

Since

$$X = X_C + X_Z, \tag{2}$$

we have

$$E(X) = E(X_C) + E(X_Z). \tag{3}$$

The distribution functions of the random variables X_C, X_Z take the forms

$$F_{X_C}(x) = \begin{cases} F_X(x), & \text{for } x < \alpha \\ 1, & \text{for } x \geq \alpha \end{cases}, \qquad F_{X_Z}(x) = \begin{cases} 0, & \text{for } x < 0 \\ F_X(x+\alpha), & \text{for } x \geq 0 \end{cases}. \tag{4}$$

As we can see, the distribution functions F_{X_C}, F_{X_Z} are not continuous, and so neither are the random variables X_C, X_Z. Therefore the probability density functions f_{X_C}, f_{X_Z} do not exist even though the random variable *X* is continuous.

Calculation of the moments of the random variables X_C, X_Z via the distribution functions F_{X_C}, F_{X_Z} is more difficult and it would require the use of a more generalised integral, for example the Stieltjes integral (see [1], p. 212 - 217). For this reason, next we will now express the moments of the random variables X_C, X_Z using a probability density function *f* or a distribution function *F* of the random variable *X*. We will assume that the density function *f* takes nonnegative values only on the interval $\langle 0; \infty)$ [or on the interval $\langle b; \infty)$, where $b > 0$] and zero otherwise. Moreover we assume that there is no interval (c, d), $c > 0$, $d > c$ such that $f(x) = 0$ for all $x \in (c, d)$ and a distribution function *F* takes the value 0 only for $x \in (-\infty, 0\rangle$. Similarly in the case if *f* takes nonnegative values only in an interval $\langle b; \infty)$.

From (1)[2], it follows that:

$$E(X_C) = \int_0^{\alpha} x f(x)\,\mathrm{d}x + \alpha(1 - F(\alpha)) = \alpha - \int_0^{\alpha} (\alpha - x) f(x)\,\mathrm{d}x = \alpha - \int_0^{\alpha} F(x)\,\mathrm{d}x, \tag{5}$$

[2] All numerical computations, theoretical derivations of formulae and graphical interpretations were realised using open source system MAXIMA.

$$E(X_Z)=\int_{\alpha}^{\infty}(x-\alpha)f(x)\,\mathrm{d}x=\int_{\alpha}^{\infty}xf(x)\,\mathrm{d}x-\alpha(1-F(\alpha))=E(X)-\alpha+\int_{0}^{\alpha}F(x)\,\mathrm{d}x, \qquad (6)$$

where

$E(X_C)$ is expected amount of the loss of the cedant,

$E(X_Z)$ is expected amount of the loss of the reinsurer.

From (5), it follows that $E(X_C)<\alpha$, namely by taking the value $\int_0^{\alpha}F(x)\,\mathrm{d}x$. Similarly, it is possible to express the initial moments of the random variables X_C, X_Z of higher order

$$E\left(X_C^k\right)=\int_0^{\alpha}x^k f(x)\,\mathrm{d}x+\alpha^k(1-F(\alpha)),\quad E\left(X_Z^k\right)=\int_{\alpha}^{\infty}(x-\alpha)^k f(x)\,\mathrm{d}x. \qquad (7)$$

2.1 The shape of some numerical characteristics depending on the priority α

Since the random variables X_C, X_Z are functions of the priority α of the cedant, in the what follows we shall use instead of $E(X_C(\alpha))$, $E(X_Z(\alpha))$, $D(X_C(\alpha))$, $D(X_Z(\alpha))$, $\sigma(X_C(\alpha))$, $\sigma(X_Z(\alpha))$, $cv(X_C(\alpha))$, $cv(X_Z(\alpha))$ the simpler expressions $EX_C(\alpha)$, $EX_Z(\alpha)$, $DX_C(\alpha)$, $DX_Z(\alpha)$, $\sigma X_C(\alpha)$, $\sigma X_Z(\alpha)$, $cvX_C(\alpha)$, $cvX_Z(\alpha)$, where $E(.)$, $D(.)$, $\sigma(.)$, $cv(.)$ are the mean, variance, standard deviation and coefficient of variation respectively.

2.1.1 Monotonicity of EX_C(α)

Theorem 1. a) *The function* $EX_C(\alpha)$ *is increasing and concave.*

b) *The function* $EX_Z(\alpha)$ *is decreasing and convex.*

Proof: From (5) and (6) we have

$$\frac{\mathrm{d}EX_C(\alpha)}{\mathrm{d}\alpha}=\left(\alpha-\int_0^{\alpha}F(x)\,\mathrm{d}x\right)'=1-F(\alpha)>0,\quad \frac{\mathrm{d}EX_Z(\alpha)}{\mathrm{d}\alpha}=F(\alpha)-1<0, \qquad (8)$$

$$\frac{\mathrm{d}^2EX_C(\alpha)}{\mathrm{d}\alpha^2}=(1-F(\alpha))'=-f(\alpha)<0,\quad \frac{\mathrm{d}^2EX_C(\alpha)}{\mathrm{d}\alpha^2}=-f(\alpha)<0, \qquad (9)$$

which proves the type of monotonicity of $EX_C(\alpha)$ and $EX_Z(\alpha)$ dependent on α, see Diagram 1.

Let us comment on the computation for a random variable X_C. The expected value of the insurance payment of the cedant is increasing (decreasing) with increasing (decreasing) priority α. This statement, of course, is not surprising from an actuarial point of view. However, the relations (8) and (9) and the ensuing shape of expected insurance payments gives us more precise information about the increase in the expected insurance payment with increasing priority α and about the instantaneous rate of change of the expected insurance payment. This instantaneous rate at the point $\alpha = \alpha_0$ is equal to the tail value of the density function *f* of the individual insurance payment at point $\alpha = \alpha_0$.

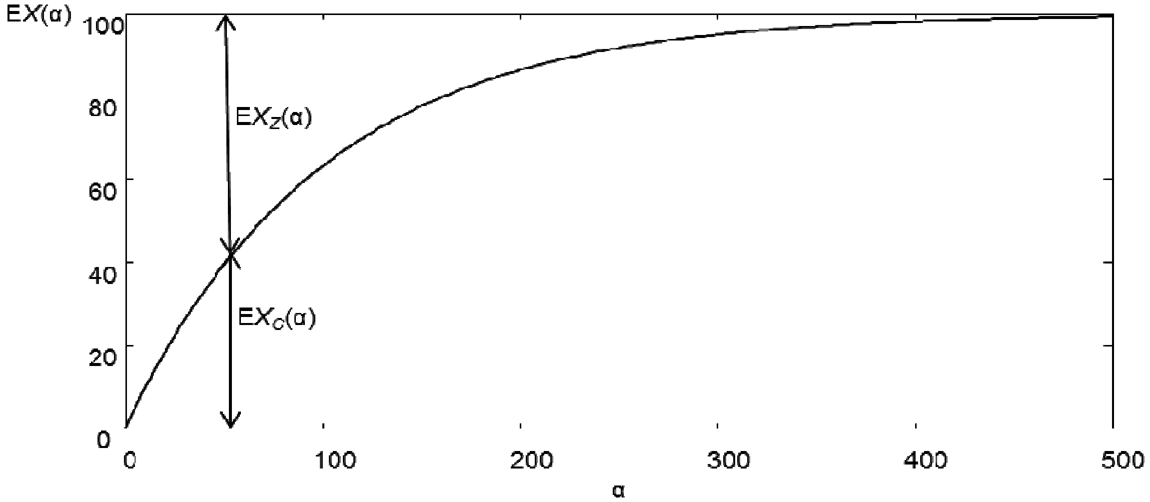
From equation (5) and the shape of EX_C(α) (Diagram 1) it follows that:

$$\lim_{\alpha\to\infty}EX_C(\alpha)=E(X),$$

$$\lim_{x\to 0^+}\frac{EX_C(\alpha)}{\mathrm{d}\alpha}=1. \tag{10}$$

From equations (8) and (10) it follows that for $\alpha > 0$ the instantaneous rate of change of the expected insurance payment of the cedant is less than 1 and that it decreases with increasing α, just as the marginal expected insurance payment does

Diagram 1: *The shape of the expected value of insurance payments of the cedant in depending on the priority α*



2.1.2 Monotonicity of D*XC*(α)

In general, studying the monotonicity of the variance $DX_C(\alpha)$ of an insurance payment of the cedant depending on the priority α is more complicated than studying the monotonicity of the expected value of the insurance payment.

We have that

$$\frac{\mathrm{d}DX_C(\alpha)}{\mathrm{d}\alpha}=\frac{\mathrm{d}EX_C^2(\alpha)}{\mathrm{d}\alpha}-\frac{\mathrm{d}E^2X_C}{\mathrm{d}\alpha}.$$

and after reducing we get

$$\frac{\mathrm{d}DX_C(\alpha)}{\mathrm{d}\alpha}=2\big(1-F(\alpha)\big)\int_0^{\alpha}F(t)\mathrm{d}t>0\,. \tag{11}$$

Equation (11) shows that the function $DX_C(\alpha)$ is increasing on the interval $(0;\infty)$.

Differentiating (11) gives

$$\frac{\mathrm{d}^2DX_C(\alpha)}{\mathrm{d}\alpha^2}=2\left[F(\alpha)\big(1-F(\alpha)\big)-f(\alpha)\int_0^{\alpha}F(t)\mathrm{d}t\right]. \tag{12}$$

The equation

$$F(\alpha)\big(1-F(\alpha)\big)-f(\alpha)\int_0^{\alpha}F(t)\mathrm{d}t=0 \tag{13}$$

is an integral-functional equation with the unknown variable in the upper limit of integration. To find the intervals of concavity and convexity of the function $DX_C(\alpha)$ we need to find the solution of the equation (13). It is easy to see that $\alpha = 0$ is the solution of the equation (13).

First we prove that the function $DX_C(\alpha)$ is convex in a small incomplete right neighbourhood of the point $\alpha = 0$. It is equivalent to prove that, in this neighbourhood, the following inequality holds

$$F(\alpha)\big(1-F(\alpha)\big)-f(\alpha)\int_0^{\alpha}F(t)\mathrm{d}t>0\,.$$

Obviously, in the mentioned incomplete right neighbourhood $F(\alpha)\big(1-F(\alpha)\big)>0$ and $f(\alpha)\int_0^{\alpha}F(t)\mathrm{d}t>0$. If it were not the case, there would have to be a point α', where α' > 0 for which $F(\alpha')=0$ or $f(\alpha')=0$ which contradicts our assumptions.

Let us calculate

$$L=\lim_{\alpha\to 0^+}\frac{F(\alpha)\big(1-F(\alpha)\big)}{f(\alpha)\int_0^{\alpha}F(t)dt}=\lim_{\alpha\to 0^+}\big(1-F(\alpha)\big)\cdot\lim_{\alpha\to 0^+}\frac{F(\alpha)}{f(\alpha)\int_0^{\alpha}F(t)dt}=\lim_{\alpha\to 0^+}\frac{F(\alpha)}{f(\alpha)\int_0^{\alpha}F(t)dt},$$

if $\lim_{\alpha\to 0^+}f(\alpha)>0$ then $L=\lim_{\alpha\to 0^+}\frac{1}{f(\alpha)}\lim_{\alpha\to 0^+}\frac{f(\alpha)}{F(\alpha)}=+\infty$,

if $\lim_{\alpha\to 0^+}f(\alpha)=0$ and $f'_+(\alpha)>0$, then

$$L=\lim_{\alpha\to 0^+}\frac{f(\alpha)}{f'(\alpha)\int_0^{\alpha}F(t)dt+f(\alpha)F(\alpha)}=\lim_{\alpha\to 0^+}\frac{1}{f'(\alpha)\frac{\int_0^{\alpha}F(t)\mathrm{d}t}{f(\alpha)}+F(\alpha)}=$$

$$=\frac{1}{\lim_{\alpha\to 0^+}f'(\alpha)\cdot\frac{\int_0^{\alpha}F(t)\mathrm{d}t}{f(\alpha)}+\lim_{\alpha\to 0^+}F(\alpha)}=\frac{1}{\lim_{\alpha\to 0^+}F(\alpha)+\lim_{\alpha\to 0^+}F(\alpha)}=\frac{1}{0^+ +0^+}=+\infty,$$

Let us further assume that

$$\lim_{\alpha\to 0^+}f(\alpha)=0 \text{ and } \lim_{\alpha\to 0^+}f'(\alpha)=0$$

and that the distribution function F is right differentiable at all levels at the point $x = 0$ and differentiable at all levels on some interval (0; δ), δ > 0, whereby these derivatives are bounded. Then there must exist a smallest natural number k, for which we have

$$\lim_{\alpha\to 0^+}f^{(k)}(\alpha)\neq 0\,.$$

Otherwise the Taylor expansion of F would be identically equal to zero, which contradicts our assumption concerning the function F.

Then

$$\lim_{\alpha\to 0^+}\frac{\int_0^\alpha F(t)\mathrm{d}t}{f(\alpha)}=\lim_{\alpha\to 0^+}\frac{F(\alpha)}{f'(\alpha)}=\lim_{\alpha\to 0^+}\frac{\frac{f^{(k)}(0)\alpha^{k+1}}{(k+1)!}+\frac{f^{(k+1)}(0)\alpha^{k+2}}{(k+2)!}+\ldots}{\frac{f^{(k)}(0)\alpha^{k-1}}{(k-1)!}+\frac{f^{(k+1)}(0)\alpha^{k}}{k!}+\ldots}=$$

$$=\lim_{\alpha\to 0^+}\alpha^2\cdot\lim_{\alpha\to 0^+}\frac{\frac{f^{(k)}(0)}{(k+1)!}+\frac{f^{(k+1)}(0)\alpha}{(k+2)!}+\ldots}{\frac{f^{(k)}(0)}{(k-1)!}+\frac{f^{(k+1)}(0)\alpha}{k!}+\ldots}=0$$

More specifically $\alpha\to 0^+ \;\Rightarrow\; \frac{F(\alpha)}{f(\alpha)}\to 0^+$.

Similarly $\alpha\to 0^+ \;\Rightarrow\; f'(\alpha)\to 0^+$.

From which we have $\alpha\to 0^+ \;\Rightarrow\; L\to\infty$

which proves that the function $y=F(\alpha)\left(1-F(\alpha)\right)$ from the point $\alpha = 0$ increases „significantly“ faster than the function $y=f(\alpha)\int_0^\alpha F(t)\mathrm{d}t$. It follows that in a small right incomplete neighbourhood of α = 0 we have $F(\alpha)\left(1-F(\alpha)\right)> f(\alpha)\int_0^\alpha F(t)\mathrm{d}t$, which proves that the function $DX_C(\alpha)$ is convex in this neighbourhood.

In general, equation (13) may have more than one positive solution. See for example the case where the probability density function f takes the form

$$f(x)=\frac{x(\cos x+1)}{0.67659458962407(x^2+1)^2}\ \text{ for } x\geq 0 \text{ and } f(x)=0 \text{ otherwise.}$$

The above probability density function represents the degenerate case of probability density functions which are not used in actuarial practice and theory. We presume that many density functions used in actuarial practice and theory, have the property that equation (13) has only one positive solution. This means that the function $DX_C(\alpha)$ has only one point of inflexion.

For example:

- for the exponential distribution with the distribution function $F(x)=1-\exp(-\beta x),\ \beta>0$, the existence and uniqueness of the positive solution of equation (13) is equivalent to the existence and uniqueness of the positive solution of the transcendent equation $\mathrm{e}^{-\beta\alpha}+\frac{\beta}{2}\alpha-1=0$, which is easy to prove,
- for a Pareto distribution with the distribution function $F(x)=1-\left(\frac{b}{x}\right)^a,\ a>0,\ b>0,\ x\geq b$ (also called the European version of the definition of the Pareto distribution) the existence and uniqueness of the positive solution of equation (13) is equivalent to the existence and uniqueness of the positive solution of the algebraic equation $(a-1)^2x^a-a^2bx^{a-1}+(2a-1)b^a=0$, which is also easy to prove.

Let α^* be a solution of equation (13). Applying the mean value theorem of integral calculus we have

$$F(\alpha^*)(1-F(\alpha^*))-f(\alpha^*)F(\hat{\alpha})\alpha^*=0, \tag{14}$$

where $F(\hat{\alpha})<F(\alpha).$ We then get an estimate for α^*

$$1-F(\alpha^*)<\alpha^* f(\alpha^*).$$

For example for the exponential distribution this is $\alpha^*>\frac{1}{\beta}$.

The critical point for deciding whether to decrease or to increase the priority α can be just the inflexion point α^*, in the neighbourhood of which the variance changes at the highest rate.

2.1.3 Monotonicity of $cvX_C(\alpha)$ the coefficient of variation of the cedant's insurance payment, dependent on the priority α

After calculation we have

$$\frac{\mathrm{d}cvX_C(\alpha)}{\mathrm{d}\alpha}=\frac{1-F(\alpha)}{\sigma X_C(\alpha)E^2X_C(\alpha)}\int_0^\alpha x(\alpha-x)f(x)dx>0$$

or

$$\frac{\mathrm{d}cvX_C(\alpha)}{\mathrm{d}\alpha}=\frac{(1-F(\alpha))\int_0^\alpha(a-x)xf(x)\mathrm{d}x}{\left(\int_0^\alpha F(x)\mathrm{d}x-\alpha\right)^2\sqrt{(2a-1)\int_0^\alpha F(x)\mathrm{d}x-2\int_0^\alpha xF(x)\mathrm{d}x}}>0.$$

In both cases, we can see that $\frac{\mathrm{d}cvX_C(\alpha)}{\mathrm{d}\alpha}>0$. This means, that the function $cvX_C(\alpha)$ is increasing in the interval $(0;\infty)$. Given the complexity of calculation we will not consider the type of monotonicity. On the other hand, for a "non-degenerative" distribution, $\lim_{\alpha\to\infty}cvX_C(\alpha)=cv(X)$ and $cvX_C(\alpha)$ will be continuous, increasing and differentiable functions on the interval $(0;\infty)$. It follows that there exists a point α', α' > 0 such that the function $cvX_C(\alpha)$ is concave on the interval $(\alpha';\infty)$.

Example 1. Let individual insurance payments have the Pareto distribution Pa(3,10) (on this occasion we consider the American version of the definition of the Pareto distribution with the distribution function $F(x)=1-\left(\frac{b}{b+x}\right)^a$). Let us examine the behaviour of the monotonicity of the expected value, variation and coefficient of variation dependant on the priority α.

*Solution***.** We have

$$f(x)=\frac{3\cdot10^3}{(10+x)^4},\quad F(x)=1-\left(\frac{10}{10+x}\right)^3,$$

$$E(X)=\frac{10}{2}=5,\quad D(X)=\frac{3\cdot10^2}{2^2\cdot1}=75.$$

After calculating in accordance with (5) we get

$$EX_C(\alpha) = \frac{5\alpha(\alpha+20)}{(\alpha+10)^2}.$$

The expected value $EX_C(\alpha)$ of the insurance payment of the cedant is increasing with increasing priority α (see Diagram 1) and asymptotically tends to the value $E(X)$. Every increment of α by 1, with an increasing α, causes a slower increase in the expected value of the cedant's insurance payments.

It is easy to derive that

$$DX_C(\alpha) = \frac{25\alpha^3(3\alpha+40)}{(\alpha+10)^4},$$

and similarly that

$$cvX_C(\alpha) = \frac{\sqrt{\alpha^3(3\alpha+400)}}{\alpha(\alpha+20)}.$$

It is easy to see that $\lim\limits_{\alpha\to\infty} DX_C(\alpha) = D(X) = 75$ and $\lim\limits_{\alpha\to\infty} cvX_C(\alpha) = cv(X) = \sqrt{3}.$

Both of the characteristics $DX_C(\alpha)$ and $cvX_C(\alpha)$ are increasing with the increasing priority α. It is possible to prove that while $cvX_C(\alpha)$ is increasing along a concave curve, the monotonicity of $DX_C(\alpha)$ is more complex. DX_C is increasing along a convex curve to the point α = 9.0587, then increasing more slowly along a concave curve.

These characteristics can help the cedant's decision making when setting the priority α.

3 Analysis of excess of loss reinsurance of individual risks with an upper limit

Let α_1 be the priority of the cedant and let α_2 be the upper limit of the insurance payment of the reinsurer (reinsurers), above which the reinsurer pays only an amount equal to the layer $\alpha_2 - \alpha_1$. We define the random variables

$$X_C = \begin{cases} X, & \text{pre } X \le \alpha_1 \\ \alpha_1, & \text{pre } \alpha_1 < X \le \alpha_2 \\ X-(\alpha_2-\alpha_1), & \text{pre } X > \alpha_2 \end{cases}, \quad X_Z = \begin{cases} 0, & \text{pre } X \le \alpha_1 \\ X-\alpha_1, & \text{pre } \alpha_1 < X \le \alpha_2 \\ \alpha_2-\alpha_1, & \text{pre } X > \alpha_2, \end{cases} \tag{15}$$

Note that if $\alpha_1 \to \alpha_2^-$, or $\alpha_2 \to \alpha_1^+$,we get the case without reinsurance.

In practice, we can encounter the following problem: how should the limit α_2 change when α_1 is changed, or conversely, so that some predefined conditions are met, for example not to change the expected value of the insurance payment.

3.1 Analysis of the monotonicity of the expected value of the insurance payment given the limits α_1, α_2

Obviously,

$$E(X_C) = \int_0^{\alpha_1} xf(x)\mathrm{d}x + \alpha_1\left(F(\alpha_2)-F(\alpha_1)\right) + (\alpha_1-\alpha_2)(1-F(\alpha_2)) + \int_{\alpha_2}^{\infty} xf(x)\mathrm{d}x$$

$$E(X_Z) = \int_{\alpha_1}^{\alpha_2} (x-\alpha_1)f(x)dx + (\alpha_2-\alpha_1)(1-F(\alpha_2))\mathrm{d}x$$

$$\frac{\partial EX_C(\alpha_1,\alpha_2)}{\partial \alpha_1} = 1 - F(\alpha_1) > 0, \quad \frac{\partial^2 EX_C(\alpha_1,\alpha_2)}{\partial \alpha_1^2} = -f(\alpha_1) < 0, \tag{16}$$

$$\frac{\partial EX_C(\alpha_1,\alpha_2)}{\partial \alpha_2} = F(\alpha_2) - 1 < 0, \quad \frac{\partial^2 EX_C(\alpha_1,\alpha_2)}{\partial \alpha_2^2} = f(\alpha_2) > 0 \tag{17}$$

The first simple interpretation of the above equations is: for a constant value of α_2, the partial function $EX_{C_{\alpha_2=\text{konšt.}}}(\alpha_1)$ is increasing with increasing α_1 along a concave curve, and for constant α_1, the function $EX_{C_{\alpha_1=\text{konšt.}}}(\alpha_2)$ is decreasing along a convex curve.

A second interpretation is associated with the derivative of the function in the direction of a vector. Let us calculate the vector derivative for a vector (1, *k*), i. e. appropriate base vector $\mathbf{I} = \left(\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\right)$.

$$\frac{dEX_C(\alpha_1,\alpha_2)}{d\,\mathbf{I}} = gradEX_P(\alpha_1,\alpha_2)\cdot\mathbf{I} = \frac{\partial EX_C(\alpha_1,\alpha_2)}{\partial \alpha_1}\frac{1}{\sqrt{1+k^2}} + \frac{\partial EX_C(\alpha_1,\alpha_2)}{d\alpha_2}\frac{k}{\sqrt{1+k^2}}.$$

We ask, for which *k*, given fixed α_1, α_2, will the derivative become negative. This means that with changes in the values of α_1, α_2 in the direction of vector **I**, the function $EX_C(\alpha_1,\alpha_2)$ will be already decreasing (in a neighbourhood of point $[\alpha_1,\alpha_2]$.

This requirement corresponds to the condition

$$\frac{\partial EX_C(\alpha_1,\alpha_2)}{\partial \alpha_1}\frac{1}{\sqrt{1+k^2}} + \frac{\partial EX_C(\alpha_1,\alpha_2)}{d\alpha_2}\frac{k}{\sqrt{1+k^2}} < 0. \tag{18}$$

From (16), (17), (18) we have

$$k > \frac{1-F(\alpha_1)}{1-F(\alpha_2)}. \tag{19}$$

Given that $\frac{1-F(\alpha_1)}{1-F(\alpha_2)} > 1$, it is also true that $k > 1$. By the way, notice that the right hand side of equation (19) is the ratio of the "tail sizes" of the density probability function with origins at α_1 a α_2.

It follows from (19) that for:

- the Pareto distribution with the American version of the definition we will have

$$k > \left(\frac{b+\alpha_2}{b+\alpha_1}\right)^a, \tag{20}$$

- and for the exponential distribution

$$k > e^{\beta(\alpha_2-\alpha_1)}.$$

Example 2. Let us suppose that individual claims have a Pareto distribution, Pa(3,1000) and $\alpha_1 = 100$ is the priority for WXL/R reinsurance and $\alpha_2 = 500$ is the upper limit of the reinsurer. It is expected that in the near future the priority α_1 will increase. The question arises - how should the upper limit α_2 change so that the expected insurance payments would not increase?

Solution. From equation (20) we have that

$$k > \left(\frac{1000+500}{1000+100}\right)^3 = 2{,}53\,.$$

Considering that the reinsurer layer is usually an integral multiple of the priority of the insurer, we shall choose the next higher integer i.e. $k = 3$.

Then

$$\frac{dEX_C(\alpha_1,\alpha_2)}{d\,\mathbf{I}} = \frac{1}{\sqrt{1+3^2}}\big(3F(500) - F(100) + 1 - 3\big) = -0{,}0435\,.$$

At the point $[\alpha_1, \alpha_2] = [100, 500]$ the function $EX_C(\alpha_1,\alpha_2)$ has a negative derivative in the direction of vector **I.** That means that if we increase α_1, α_2 from the point [100, 500] in the direction of the vector **I,** the function $EX_C(\alpha_1,\alpha_2)$ will be decreasing. For example, if α_1 increases by 10, we would have to increase α_2 to a value of 30, so that this change decreased the expected insurance payment.

By calculating we would find that $EX_C(100,500) = 309$ and $EX_C(110,530) = 307{,}782$. The expected insurance payment would indeed decrease, but only by a little amount, because the derivative in the direction of vector **I** is negative and is close to zero.

3.2 Analysis of the monotonicity of the variance of the cedant's payments given limits α_1, α_2

Let us calculate

$$DX_C(\alpha_1,\alpha_2) = EX_C^2(\alpha_1,\alpha_2) - E^2X_C(\alpha_1,\alpha_2) =$$
$$= \int_0^{\alpha_1} x^2 f(x)dx + \alpha_1^2\big(F(\alpha_2) - F(\alpha_1)\big) + \int_{\alpha_2}^{\infty}(x-\alpha_2+\alpha_1)^2 f(x)dx - E^2X_C(\alpha_1,\alpha_2).$$

Differentiating $DX_C(\alpha_1,\alpha_2)$ with respect to α_1 we have

$$\frac{\partial DX_C(\alpha_1,\alpha_2)}{\partial\alpha_1} = 2\left(\big(1-F(\alpha_1)\big)\int_0^{\alpha_1}(\alpha_1 - x)f(x)dx + F(\alpha_1)\int_{\alpha_2}^{\infty}(x-\alpha_2)f(x)dx\right) > 0, \tag{20}$$

because both of the elements, in the large bracket are positive. So by increasing α_1 (up to the value of α_2) the variance will increase, and decreasing α_1 down to 0, the variance will decrease.

It is possible to show that

$$\frac{\partial^2 DX_C(\alpha_1,\alpha_2)}{\partial\alpha_1^2} = 2\big[F(\alpha_1)\big(1-F(\alpha_1)\big) + f(\alpha_1)\big(EX_C(\alpha_1,\alpha_2) - \alpha_1\big)\big], \quad \alpha_1 \in (0,\alpha_2) \tag{21}$$

The type of monotonicity depends on the sign of the right-hand side of equation (21) for a fixed α_2 and for $\alpha_1 \in (0,\alpha_2)$. In general, it may take only positive, or negative, or positive and negative values.

Similarly, as before, it is possible to derive

$$\frac{\partial DX_C(\alpha_1,\alpha_2)}{\partial\alpha_2} = -2\left(\big(1-F(\alpha_2)\big)\int_0^{\alpha_1}(\alpha_1 - x)f(x)dx + F(\alpha_2)\int_{\alpha_2}^{\infty}(x-\alpha_2)f(x)dx\right) < 0,$$

The last equation demonstrates the fact that by increasing the reinsurer's upper limit α_2 the variance of the cedant's payments will decrease, and that by decreasing the reinsurer's upper limit (down to the value α_1), the variance will increase (for an unchanged value of the priority α_1). The type of monotonicity is given by

$$\frac{\partial^2 DX_C(\alpha_1,\alpha_2)}{\partial\alpha_2^2} = 2\left[F(\alpha_2)\left(1-F(\alpha_2)\right)+f(\alpha_2)\left(\alpha_1 - EX_C(\alpha_1,\alpha_2)\right)\right], \tag{22}$$

which for $\alpha_2 \in (\alpha_1, \infty)$ can take, in general, either positive or negative values.

4 Conclusion

Analysis of the behaviour of measures of variability and location of the insurance payments of a cedant and reinsurer dependant on changes of priority and upper limit of insurance payments may be useful for the optimal management of risks for the cedant and the reinsurer. In the article we have studied only the case of an individual risk. Similarly, one can analyze the case of a portfolio of risks that are independent and identically distributed.

References

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Summary

Závislosť zmien niektorých mier variability a polohy poistných plnení cedanta od zmeny priority v neproporcionálnom zaistení

Príspevok sa zaoberá štúdiom závislosti zmien niektorých mier variability a polohy poistného plnenia cedanta od zmeny priority v zaistení WXL/R bez horného a s horným ohraničením poistného plnenia zaisťovateľa. Okrem toho analyzuje aj prípad, ako reagovať na zmenu priority (resp. horného ohraničenia) zmenou horného ohraničenia poistného plnenia (resp. zmenou priority) tak, aby sa očakávané poistné plnenie cedanta nezvýšilo.

Tento príslevok vznikol v rámci riešenia projektu VEGA 1/0931/11: Analýza a modelovanie rizík v zmysle kvantitatívnych štúdií QIS5 projektu SOLVENCY II.