The application of dynamic methods into yield curve modeling

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Abstract

In countries with well developed inter-bank market the parsimonious models play an important role in the monetary policy. For several years central banks utilise different objective functions to determine whether the model fit the market data. To show which model is appropriate to local market data the dynamic methods were applied. They cover the functional form of the implied forward rate volatility and correlation surface of the implied forward rate. With the assumption of rational expectation the implied forward rate is particularly interesting for central banks especially if its length matches with the maturity of the central bank's key interest rate. The paper shows the appropriate shapes of the form and surface and points the objective function for yield curve modelling in Polish market (based on WIBOR data).

Key words

yield curve estimation, parsimonious models, dynamic modelling

JEL Classification: C53, C92, E43, E58

1. Introduction

Following the definition suggested by Nawalkha, Soto, Beliaeva (2004), a term structure of interest rates gives the relationship between the yield of the investment with the same credit quality but different term to maturity. There are plenty of methods, widely described by James, Weber (2000) which let create a yield curve but typically it is built with a set of liquid and common assets; every instrument can be considered as a portfolio of zero-coupon bonds (with the maturities adequate to the payment dates). Price of the zero-coupon bond is expressed as a discount factor which represents the relationship between the spot rate and the forward one (Shiller et al. 1983 and 1990, Fama 1976).

Because financial markets offer only discrete data, one of the most important problems to be solved is the model selection for fitting the data. In countries with a well developed debt market (like European one) central banks use parsimonious models coming from works of Nelson-Siegel (1987) and Svensson (1994). Because the parsimonious models let build the implied forward rate directly the implied forward rate plays a crucial role in yield curve analysis and in a process to choose the best fitting criteria. In this paper the application of two dynamic methods will be presented. First one is connected with volatility of the implied forward rates and as Wu (2009, pp. 197-198) states the volatility function is hump-shaped. The second comes from derivative pricing and is based on de-correlation effect which is characteristic for correlation surface. The farther apart the two forward rates are the less there should expect them to move similarly.

The paper is structured as follows: Section 2 provides a general overview of the term structure modelling based on parsimonious model – both Nelson-Siegel and Svensson,

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Section 3 shows the methodology of implied forward rate construction and dynamic methods Section 4 is a final part which covers concluding remarks.

2. A yield curve construction

Generally a term structure is typically built with a set of liquid and common assets; based on typically used models. In monetary policy when goodness of fit is not so important as forecasting power, the most appropriate are parsimonious models (based on Nelson-Siegel or Svensson) and the cubic splines ones. These models are generally used because they are able to create four most popular shapes: positive, negative, flat and humped. The problem arises when one should show the criteria which let choose the best model. Because goodness of fit is not the best solution there is a need to find other criteria which help to achieve the best approximation.

For the further analysis we utilise the Nelson-Siegel model with four parameters (NS_) and the Svensson model with six parameters (Sv_). A set of parameters could be estimated by three different objective functions covering mean square errors minimising between:

- market prices and theoretical ones: $\frac{\sum_{l=1}^{k}(P_l - \overline{P_l})^2}{k} \to \min$, (NS_P, Sv_P);

- market and theoretical yields: $\frac{\sum_{l=1}^{k}(i_l - \overline{i_l})^2}{k} \to \min$, (NS_Y, Sv_Y);

- prices divided by duration $\frac{\sum_{l=1}^{k}(P_l - \overline{P_l})^2}{MD_l} \to \min$, (NS_P/D, Sv_P/D);

where: $P_l - \overline{P_l}$ – a price error of l-th bond, $i_l - \overline{i_l}$ – a yield error of *l*-th bond, MD_l - modified duration, *k* – number of bonds

3. The implied forward rate

Let τ be a number of the day for which the asset prices (data) for yield modelling are taken. Using the parsimonious models for the given day τ it is possible to create the instantaneous forward rate $f_\tau(s, s+\Delta s)$ with finite-length tenors $\Delta s > 0$, the expiry at time s and maturing at time $s+\Delta s$.

The assumed tenors should be similar to the length of open market operation (OMO) or to the similar transaction which is important from monetary policy point of view. For the National Bank of Poland (NBP) it is 7 days the same as in European Central Bank's case. Dealing with the general form for the implied forward rate $f_\tau(s, s+\Delta s) = \frac{1}{\Delta s} \cdot \ln \frac{\delta(\tau, s)}{\delta(\tau, s+\Delta s)}$ and assuming that the tenor is $\Delta s = \frac{7}{365}$ (yearly) 7-days implied forward is given by:

$$f_\tau(s, s+\frac{7}{365}) = \frac{365}{7} \cdot \ln \frac{\delta(\tau, s)}{\delta(\tau, s+\frac{7}{365})}, \tag{1}$$

where: $\delta(\tau, s)$ - discount factor,

$f_\tau(s, s+\frac{7}{365})$ - the implied 7-days forward rate,

τ - the moment when the forward transaction is calculated,

s - the expiring date.

For the further explanation the example of seventeen implied forward rates were taken into account (in weekly intervals) based on WIBOR rate (Warsaw Inter Bank Offer Rate, the fixing rate from Polish inter-bank money market). Therefore the set has the following form:

$$\left\{ f_\tau(s, s+\frac{7}{365}) \right\}_{s=\frac{7}{365},\frac{14}{365},\dots,\frac{119}{365}} . \tag{2}$$

Since the form (2) is a sequence it could be written in the form of the vector $\mathbf{f}_\tau$:

$$\mathbf{f}_\tau(s, s+\tfrac{7}{365}) = \left[f_\tau(\tfrac{7}{365}; \tfrac{14}{365}); f_\tau(\tfrac{14}{365}; \tfrac{21}{365}); \dots; f_\tau(\tfrac{119}{365}; \tfrac{126}{365}) \right]. \tag{3}$$

This vector represents the 7-days long implied forward rate calculated at time τ with the expiring date in a week, two weeks, three weeks, … and seventeen weeks. It should be stressed that there are rates with the same length (tenor) but their expiring date is different.

3.1 Volatility as the first dynamic measure

To analyse the dynamic of the sequence's elements there has to define the change of implied forward rates. Taking into account the nature of discrete data $\tau = \{1,2,\dots,504\}$, the daily change of the implied forward rate $\Delta f_\tau(s, s+\frac{7}{365})$ has the form of the logarithmic rate of return:

$$\Delta f_\tau(s, s+\tfrac{7}{365}) = \ln \frac{f_\tau(s, s+\frac{7}{365})}{f_{\tau-1}(s, s+\frac{7}{365})}. \tag{4}$$

A set of rates of returns could be noted in a vector projection:

$$\Delta\mathbf{f}_\tau(s, s+\tfrac{7}{365}) = \left[\Delta f_\tau(\tfrac{7}{365}; \tfrac{14}{365}); \Delta f_\tau(\tfrac{14}{365}; \tfrac{21}{365}); \dots; \Delta f_\tau(\tfrac{119}{365}; \tfrac{126}{365}) \right]. \tag{5}$$

The main characteristics which describes the dynamic of implied forward rate changes (the elements of vector $\Delta\mathbf{f}_\tau(s; s+\frac{7}{365})$ where $s = \frac{7}{365}; \frac{14}{365}; \dots; \frac{119}{365}$) is volatility. To measure it the standard deviation was utilised. To show it in yearly terms the notation of Fabozzi, Mann, Choudhry, (2003, pp. 185) was applied:

$$\sigma(s; s+\tfrac{7}{365}) = \sqrt{\frac{250}{503} \sum_{\tau=2}^{504} \left(\Delta f_\tau(s; s+\tfrac{7}{365}) - \overline{\Delta f_\tau}(s; s+\tfrac{7}{365}) \right)^2} , \tag{6}$$

where: $\sigma(s; s+\frac{7}{365})$ - standard deviation,

$\Delta f_\tau(s; s+\frac{7}{365})$ - daily change of implied forward (4),

$\overline{\Delta f_\tau}(s; s+\frac{7}{365})$ - the average of daily change of implied forward.

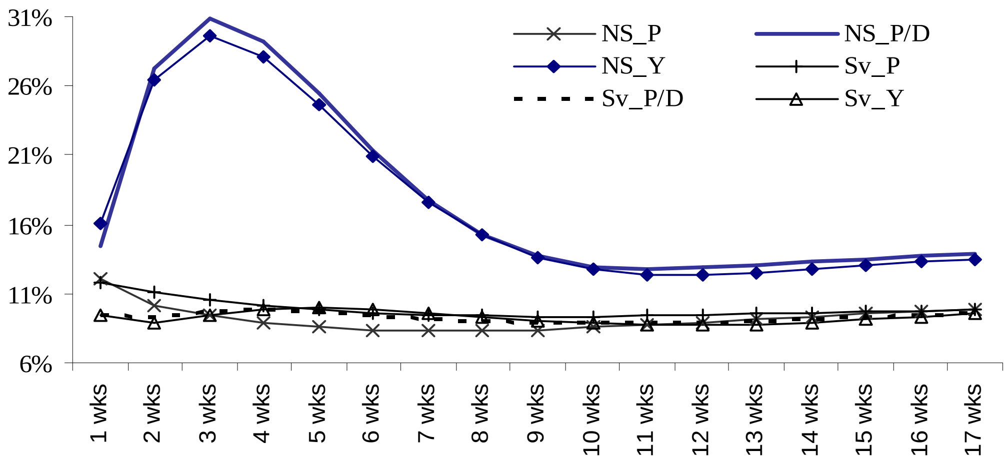
The functional form of implied volatility could be described as a relation between volatility $\sigma(s; s+\frac{7}{365})$ and the maturity $s+\frac{7}{365}$ in a form of:

$$\left\{ \sigma(s, s+\frac{7}{365}) \right\}_{s=\frac{7}{365},\frac{14}{365},\dots,\frac{119}{365}} \tag{7}$$

Rebonato (2004, pp. 672-673) states that in spite of different forms of the implied forward the most desirable is the humped-shape one. It means that the volatility of the implied forward rate changes increases as the expiration date goes to one year. Then it starts to decrease when the expiration date goes to infinity. Sometimes a decreasing shape is also noticed.

The functional form of volatility creates such kind of shape is a consequence of the implied forward rate behaviour. Rebonato (2002, p. 154-172) states that the instantaneous volatility of forward rates with a short time to expiry is expected to be low because of monetary policy which generally influences the short segment of the yield curve. The segment between one month and one year is mostly influenced by market expectations; changes in official monetary policy make to modify the market's forecasts and tend to increase forward rate volatility. The longer end of the yield curve is limited affected by market behaviour – it is mainly influence by inflation expectations and long-term policy movements. This is why the longer is the term to expiration date $(s-\tau)$ the lower is the volatility of the implied forward rates changes.

Figure 1: The implied volatility for the different estimation procedure.



The figure 1 shows how much differ the volatility for depending on the objective function (the base data, WIBOR, is the same in all six circumstances). According to figure 1 the best approximation is offered by four models: all Svensson (Sv_P, Sv_P/D, Sv_Y) and Nelson-Siegel model with objective function based on prices (NS_P/D). The rest two shows high volatility in short term.

3.2 Correlation surface as the second dynamic measure

Following the former description of the implied 7-days forward rate for seventeen different expiration dates $s=\frac{7}{365},\frac{14}{365},\ldots,\frac{119}{365}$ shown in the formula (5) as the vector: $\Delta\mathbf{f}_\tau(s,s+\frac{7}{365})=\left[\Delta f_\tau(\frac{7}{365};\frac{14}{365});\Delta f_\tau(\frac{14}{365};\frac{21}{365});\ldots;\Delta f_\tau(\frac{119}{365};\frac{126}{365})\right]$, one can modify this form by replacing $n=1,2,\ldots,17$ by $s\equiv\frac{7n}{365}$:

$$\Delta\mathbf{f}_\tau(\tfrac{7n}{365},\tfrac{7(n+1)}{365})=\left[\Delta f_\tau(\tfrac{7}{365};\tfrac{14}{365});\Delta f_\tau(\tfrac{14}{365};\tfrac{21}{365});\ldots;\Delta f_\tau(\tfrac{119}{365};\tfrac{126}{365})\right]. \tag{8}$$

Therefore there is a possibility to calculate the correlation between two different rates:

$$\rho_{n,m}\equiv\rho_{n,m}\left(\Delta f_\tau(\frac{7n}{365},\frac{7(n+1)}{365});\Delta f_\tau(\frac{7m}{365},\frac{7(m+1)}{365})\right), \tag{9}$$

where: $\rho_{n,m}$ - correlation coefficient for $n,m=1,2,\ldots,17$,

$\Delta f_\tau(\frac{7n}{365},\frac{7(n+1)}{365})$ - the change of the forward described as:

$$\Delta f_\tau(\tfrac{7n}{365},\tfrac{7(n+1)}{365}) = \ln \frac{f_\tau(\tfrac{7n}{365},\tfrac{7(n+1)}{365})}{f_{\tau-1}(\tfrac{7n}{365},\tfrac{7(n+1)}{365})} \text{ for } n = 1,2,\ldots,17 .$$

The calculated correlation coefficients $\rho_{n,m}$ create the correlation matrix $\mathbf{P}$ with seventeen rows and seventeen columns:

$$\mathbf{P} = \{\rho_{n,m}\}_{n=1,2,\ldots,17;\; m=1,2,\ldots,17} , \tag{10}$$

where: $\rho_{n,m} \equiv \rho_{n,m}\left(\Delta f_\tau(\frac{7n}{365},\frac{7(n+1)}{365});\Delta f_\tau(\frac{7m}{365},\frac{7(m+1)}{365})\right)$, for $n,m = 1,2,\ldots,17$.

The matrix $\mathbf{P}$ covers correlation coefficients $\rho_{n,m}$ should have following features (Rebonato, 2002, pp. 190-191):

[1] $\rho_{m,m} = 1$, for any m,

[2] $-1 \le \rho_{n,m} \le 1$, $n,m = 1,2,\ldots,17$,

[3] $\rho_{m,n} = \rho_{n,m}$.

Additionally for the correlation coefficient $\rho_{n,m}$ the crucial is the distance between the sequences (elements of the vector $\mathbf{\Delta f}_\tau$) which is explained in un weeks by the modulo of difference $|n-m|$. The bigger is the distance, the lower correlation is expected. This feature which is common in money market was firstly described by Brace, Gątarek, Musiela (1997, pp. 127-154) who named it as de-correlation effect and proved that it is analogous to the phenomenon observed among time series. It refers that the highest correlation is noticed between rates which dates of expiring are close (the distance $|n-m|$ is small). This property could be written as:

[4] for fixed n, $\rho_{n,n+m}$ should be a decreasing function of m.

It is worth to stress that the de-correlation process is not going to zero but to any positive number:

[5] for fixed n, $\lim_{m\to\infty} \rho_{n,n+m} = a > 0$.

Additionally, the longer is the term to the expiring date the de-correlation effect is weaker. It comes from the behaviour of market participants. The market expectations and forecasts about the forward rates in far future should not differ too much and generally are quite similar. In case of nearest future market participants have enough information to differ their expectations according to the term:

[6] for fixed m, $\rho_{n,n+m}$ should be an increasing function of n

For the correlation surface, the following description is assumed:

n wks - the expiration of $f_\tau\left(\frac{7\cdot n}{365},\frac{7(n+1)}{365}\right)$ where $n = 1,2,\ldots,17$.

The application of the correlation surface into modelling process let choose better form of approximation. If created surface fulfil [1]-[6] criteria it means that the functional form used for yield curie construction was appropriate. Any deviations suggest that one of the reasons could lies in the functional form chosen for approximation.

Figure 2: The correlation surface for Nelson-Siegel model with objective function based on prices (a), yields (b) and prices divided by duration (c).

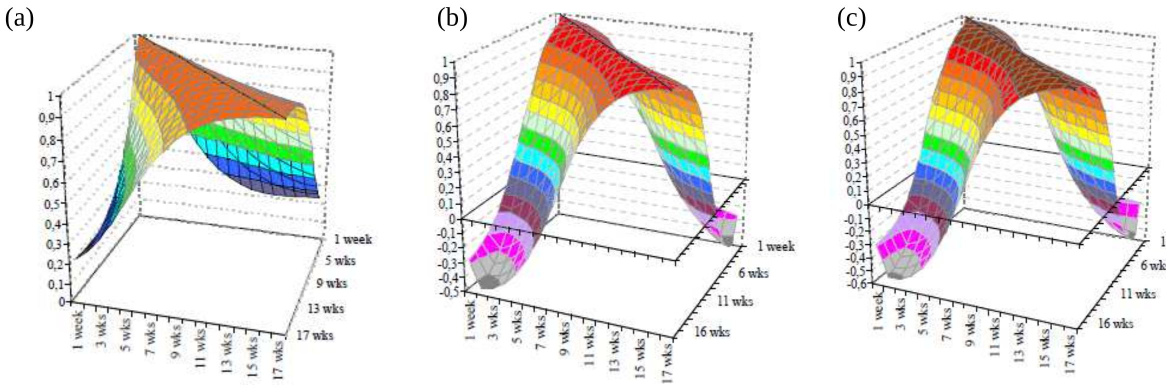
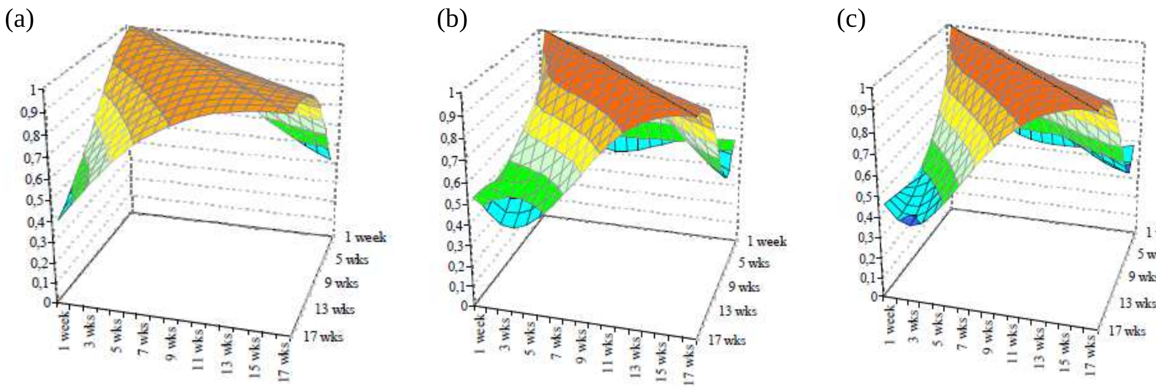


Figure 3: The correlation surface for Svensson model with objective function based on prices (a), yields (b) and prices divided by duration (c).



The figure 2 and 3 show the three-dimension surfaces which were created from the elements of the matrix $\mathbf{P}$. Two coordinates cover the expiring dates of different implied forward rates, when the third one the correlation coefficient value. The shapes let choose the best method of yield curve modelling. Following the properties [1]-[6], the best approximation which fulfil the criteria is Svensson model based on yields and prices divides by duration (Sv_Y, Sv_P/D) and Nelson-Siegel based on prices (NS_P). The rest does not fulfil the feature [5] so it can achieve the negative value.

4. Summary

In countries with well developed debt market there are parsimonious models that play an important role in term structure building process. This paper took data for inter-bank deposits (WIBOR) from 2005-2009 period. Parameters were compared every 7-days to examine the power of dynamic methods in yield curve modelling.

During the period 2005-2009 we found that the best approximation is created by the Svensson model based on yields and prices divides by duration (Sv_Y, Sv_P/D) and Nelson-Siegel based on prices (NS_P). They fulfil the optimal criteria necessary in volatility form and correlation surface.

These results should be interpreted with caution, because a small open market, sensitive to external shocks and speculative attacks (like Polish one) is too changeable to recognize results as typical. It is also too early to indicate with no doubts the best fitting method. The result we received show how wide is spectrum of implied forward rate for the same date. We should be very careful about formulating the conclusions.

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