The Influence of Extreme Claims on the Risk of Insolvency

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Abstract

In this paper, the classical risk process with light-tailed distributions (small claims) and its modifications for extreme claim amounts with heavy-tailed distributions are studied. The risk of insolvency is measured here by the probability of ruin. We present some approximations and upper bounds for the probability of ruin, and consider the conditions under which these models run correctly.

Keywords

Risk process, Cramér-Lundberg model, insolvency, ruin probability, approximations, upper bounds.

1 Introduction

Risk is a key concept to characterize future uncertain outcomes in financial sphere. The central problem of the insurance risk theory is modelling the probability distribution of total (aggregate) future claims which is than used to evaluate ruin probability (likelihood of insolvency). When the reserve of an insurer company falls below zero as a result of the last claim, we say that ruin has occurred.

By the basic actuarial principle, premiums are calculated by relying on the mean value of total claims increased by the relative safety loading (insolvency coefficient). Important decision variables are the initial capital (initial reserve) of the insurer and the risk premium influenced by the safety loading. Practical actuarial approaches often ignore complex interdependences among timing of claims, their sizes (especially extreme claim amounts) and the possibility of ruin.

In this paper, we will consider the general collective risk model with some modification,\[ R(t) = R(0) + C(t) - S(t), \] (1.1)

where \( R(t) \) is the risk reserve at time \( t \), \( R(0) = u \) the initial reserve at time \( t = 0 \). The total risk premium is denoted by \( C(t) \), and \( S(t) \) represents the total claim amount up to time \( t \). We suppose that the individual claims \( X_i \) are nonnegative independent identically distributed (iid) random variables and that they occur in random times \( T_i, i = 1, 2, 3 \ldots \). Further, we suppose that the inter-arrival (waiting) times \( W_i = T_i - T_{i-1}, i = 2, 3, \ldots, W_1 = T_1 \), are iid and the sequences \( \{X_i\} \) and \( \{W_i\} \) are independent. Denote by \( N(t) \) the total number of claims up to time \( t \), then \( \{N(t), t \geq 0\} \) is a counting process (or claims arrival process), for which we have \( N(t) = \max \{ n; T_n \leq t \} \). In general, \( \{R(t)\} \) and \( \{S(t)\} \) are random processes, and in some special cases also is \( \{C_i\} \). The total claim \( S(t) = X_1 + X_2 + \ldots + X_{N(t)} \) has a compound

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distribution

\[ F_{S(t)}(x) = P(S(t) \leq x) = P\left( \sum_{i=1}^{N(t)} X_i \leq x \right) = \sum_{n=0}^{\infty} P(N(t) = n) \cdot F^n(x), \quad (1.2) \]

where \( F^n(x) \) is the n-fold convolution of the distribution \( F \).

There exist different modifications of the general risk process, introduced in (1.1). We can consider different special distributions for \( N(t) \) and \( X_i \). For example, \( N(t) \) can be distributed Poissonial, binomial or negative binomial, and the claims \( X_i \) may have light-tailed distribution for small claims (exponential, normal, gamma), or heavy-tailed subexponential distribution for extremal claims (Pareto, Burr, Weibull, loggamma, etc.). The premium income rate \( c \) is usually constant \((C(t) = c \cdot t)\), but it can be also another non-random function of time, even random variable (as in bonus-malus systems, see [6]).

In the following Section 2 we will consider the historical special case of the general risk process, called Cramér-Lundberg process (C-L model), and will deal with some approximations and upper bounds for the probability of ruin in this case. For this purpose we define

\[ \psi(u,T) = P\{R(t) < 0, \text{ for some } t \leq T\}, \quad 0 < T < \infty, \quad u \geq 0 \quad (1.3) \]

the probability of ruin in finite time (with finite horizon), and by \( \psi(u) = \psi(u,\infty) \), \( u \geq 0 \), the probability of ruin in infinite time (with infinite horizon).

## 2 The standard Cramér-Lundberg model

The historical version of the general risk process (1.1) was introduced by Lundberg (1903) and Cramér (1930), and is given by the following **C-L conditions**:

C1) The claim arrival process \( \{N(t), t \geq 0\} \) is homogeneous Poisson process with constant intensity \( \lambda \), \( E(N_t) = \lambda t \).

C2) The claim sizes \( X_i, i = 1,2,... \) are nonnegative iid random variables with common distribution function \( F \), finite mean \( E(X_1) = \mu < \infty \) and finite variance \( D(X_1) < \infty \).

C3) The inter-arrival times \( W_i \) are independent and exponentially distributed with finite mean \( E(W_1) = 1/\lambda \).

C4) The net premium \( C_t \) up to time \( t \) is considered to be payable at constant rate \( c \) per unit time, so that \( C_t = c \cdot t \).

The risk process is now defined as \( R_t = u + c \cdot t - (X_1 + X_2 + ... + X_{N(t)}) \), and its possible realization is on the following picture.

![Risk Process Diagram](image)

The expected value of risk reserves is
Under the net profit condition, we require \( c - \lambda \mu > 0 \), more exactly \( c = (1 + \rho)\lambda \mu \), where the relative safety loading \( \rho \) is defined as
\[
\rho = \frac{c - \lambda \mu}{\lambda \mu} > 0.
\]

The initial reserve \( R(0) = u \) and the safety loading \( \rho \) are important decision parameters for the risk management. According to (1.2), the total claim amount \( S_t \) has a compound Poisson distribution given by
\[
F_{S_0}(x) = P(S(t) \leq x) = \sum_{k=1}^{\infty} \left(\frac{\lambda t}{k!}\right) e^{-\lambda t} \cdot F^*(x), \quad x \geq 0, \ t \geq 0.
\]

Under the net profit condition \( \rho > 0 \) one can show (see Embrechts [5]), that for the non-ruin probability holds
\[
1 - \psi(u) = \frac{\rho}{1 + \rho} \sum_{n=0}^{\infty} \left(\frac{1}{1 + \rho}\right)^n F_i^{**}(u), \ u \geq 0,
\]
where \( F_i(u) = \frac{1}{\mu} \int_0^\mu \overline{F}(y)dy \) is the integrated tail distribution of \( F \) and \( \overline{F}(y) = 1 - F(y), \ y > 0 \).

The relation (2.4) says, that the non-ruin probability \( 1 - \psi(u) \) represents a compound geometric distribution. This formula is the key tool for calculating ruin probabilities, but in general, it is difficult to derive explicit expressions for \( \psi(u) \) by (2.4), except the case of exponentially distributed small claims with \( F(x) = 1 - e^{-x/\mu}, \ x \geq 0, \ \mu > 0 \). In this case we get (see [9])
\[
\psi(u) = \frac{1}{1 + \rho} \cdot \exp\left\{ -\frac{\rho \cdot u}{(1 + \rho) \cdot \mu} \right\}, \quad u \geq 0.
\]

Grandell in 1991 proved (see [8]) that the exact ruin probability satisfies the following integral equation of renewal type
\[
\psi(u) = \frac{\lambda}{c} \cdot \int_0^u \psi(u - y) \overline{F}(y)dy + \frac{\lambda}{c} \cdot \int_u^\infty \overline{F}(y)dy.
\]

In general, again no explicit closed form solution to the equation (2.6) exists, except in the case where claims are mixtures of exponential distributions (see [1]). In non-exponential cases, under suitable conditions, one can obtain only some approximations to the ruin probability. The well known upper bound for the ruin probability gives the Cramér-Lundberg inequality as we can see in the following theorem.

**Theorem 2.1.** Let the net profit condition \( \rho > 0 \) hold. Assume that there exists a positive constant \( L > 0 \) (called adjustment coefficient or Lundberg coefficient), such that
\[
\frac{\lambda}{c} \cdot \int_0^\infty e^{-Lx} \overline{F}(x)dx = 1.
\]

Then for all \( u \geq 0 \) holds
\[
\psi(u) \leq e^{-Lu}.
\]
Proof. There exist several different proofs of the Lundberg inequality (2.8). The original proof is based on Wiener-Hopf method, Gerber proved it by martingale method (see [7]), but the most simply proof uses mathematical induction method (see for example [2], [9]).

Remarks.

• to calculate the Lundberg coefficient $L$ as a positive solution of (2.7), in general is rather complicated. The alternative version of Lundberg condition in terms of moment generating functions is

$$e^{-cz} \cdot M_X(z) = 1, \quad (2.9)$$

where $M_X(z) = E(e^{zX})$ is the moment generating function of individual claim $X$.

• for exponentially distributed claims with mean value $\mu$, after putting $c = (1 + \rho)\lambda\mu$, one can easy calculate that the solution of (2.9) is:

$$z = L = \rho l/(1 + \rho)\mu. \quad (2.10)$$

• when the claims are not exponentially distributed, but they have finite mean and variance, we get an approximation for safety loading using the Taylor expansion of the moment generating function of individual claims in form:

$$L \approx 2\rho E(X)/D(X). \quad (2.11)$$

• in the case when the claims are from interval $(0, M)$, we can derive the following upper or lower bounds for safety loading (see [2], [9])

$$\frac{\ln(1 + \rho)}{M} \leq L \leq \frac{2\rho E(X)}{E(X^2)} \quad (2.12)$$

Another general formula for calculation $\psi(u)$ using conditional mean value as a function of the time of first ruin, gives the following theorem (for details see [5] or [11]).

**Theorem 2.2.** For arbitrary $u \geq 0$ holds

$$\psi(u) = e^{-L\cdot u}/E(e^{-L\cdot R(T)} \mid T < \infty), \quad (2.13)$$

where $T = \min\{t; R(t) < 0, t > 0\}$ is the time of the first ruin.

### 3 Modification of Cramér-Lundberg model for large claims

When claims in the compound Poisson risk model are from heavy-tailed subexponential distributions (Pareto, Burr, Weibull or lognormal), classical techniques used to compute the probability of ruin in Section 2, are not suitable. In this case, the individual claims have not finite exponential moments ($E(e^{zX}) = \infty$, for any $z > 0$), so that the moment generating function does not exist. Under high claims, the probability of ruin is larger than in the classical case of finite moment generating function. So, to approximate $\psi(u)$ and construct some non-exponential bound for large $u$ without the Cramer-Lundberg condition, is very important problem. In this Section 3 we will present some results of this type, but at first, we define the subexponential distributions.

**Definition 3.1.**

A distribution function $F$ with support $(0, \infty)$ is called subexponential, if for all $n \geq 2$,

$$\lim_{x \to \infty} \frac{F^{*n}(x)}{F(x)} = n. \quad (3.1)$$

The class of subexponential distribution functions will be denoted as $S$. 
Remark.
In the case of subexponential distributions, the tails of the sum \( S_n = X_1 + X_2 + \ldots + X_n \) and of the maximum \( M_n = \max \{ X_1, X_2, \ldots, X_n \} \) are equivalent. The tail probabilities are
\[
P(S_n > x) = P(M_n > x), \quad n \to \infty.
\]

**Theorem 3.1.** Consider the C-L risk model with net profit condition \( \rho > 0 \) and large claims with subexponential distribution \( F_1 \in S \). Then for \( u \to \infty \)
\[
\psi(u) \approx \frac{1}{\rho} \cdot F_1(u).
\]

**Proof.** Using formula (2.4) and the definition of subexponential distribution, we have
\[
\frac{\psi(u)}{F_1(u)} = \frac{\rho}{1 + \rho} \cdot \sum_{n=0}^{\infty} (1 + \rho)^{-n} F_1^n(u), \quad u \geq 0.
\]
From this we get the approximation (3.2). \( \square \)

**Theorem 3.2.** Consider the C-L risk model with net profit condition \( \rho > 0 \) and large claims with subexponential distribution \( F_1 \in S \). Further consider that for \( X_i, \ i = 1, 2, \ldots \) the first \( m \) moments exist (i.e. for all \( k \leq m, \ E(X^k) = \int x^k dF_i(x) < \infty \)). Let there exist a constant \( k > 0 \) satisfying for \( 0 < \theta < 1 \) the conditions
\[
\int_0^{\infty} (1 + kx)^m dF_i(x) = \frac{1}{\theta} \quad \text{and} \quad P(N > n + 1) \leq \theta \cdot P(N > n).
\]
Then the following inequality holds for every \( u \geq 0 \)
\[
\psi(u) \leq (1 + ku)^{-m}.
\]

**Proof.** For the proof see Conti, 2009 [4].

**Theorem 3.3.** Consider the C-L risk model with net profit condition \( \rho > 0 \) and large claims. Assume that for any \( u > 0 \) there exists a constant \( k_0 > 0 \) so that \( \int_0^u e^{k_0 x} dF(x) = 1 + \rho \). Then for the ruin probability we obtain the following upper and lower bounds
\[
\frac{\rho \cdot \exp\{-2k_0 u\} + F(u)}{\rho + F(u)} \leq \psi(u) \leq \frac{\rho \cdot \exp\{-k_0 u\} + F(u)}{\rho + F(u)}.
\]

**Proof.** The proof one can find in Cai and Garrido, 1999 [3].

In the last decade, some results were published about so called “new worse than used” distributions which are based on stochastic ordering and are suitable to generalize the C-L inequality for the probability of ruin.

**Definition 3.1.**
A distribution $G$ supported on $[0, \infty)$ is said to be “new worse than used” (NWU), if for any $x > 0$ and $y > 0$,
\[
\overline{G}(x+y) > \overline{G}(x) \cdot \overline{G}(y).
\] (3.6)

**Theorem 3.4.** Assume that there exists an NWU distribution $G$ so that for $\rho > 0$
\[
\int_0^\infty \left( \overline{G}(x) \right)^{-1} dF(x) = 1 + \rho.
\]
Then
\[
\psi(u) \leq \overline{G}(u), \text{ for } u \geq 0.
\] (3.7)

**Proof.** For proof see Willmot and Lin, 2001 [12].

4 Conclusion

The results, presented in Sections 2 and 3, have theoretical character, but they can be useful also in practical insurance management for calculating the probability of ruin. We dealt with statistical analysis of some insurance data including extreme claims already in [10], but this time we have obtained actual real non-life insurance data and our goal is their detailed analysis with emphasis on ruin probability calculation.

**Literature**


Summary

Vplyv extrémnych poistných plnení na riziko insolventnosti

Príspevok sa zaobera s klasickým procesom rizika, kde malé poistné plnenia majú rozdelenie s „ľahkým chvostom“, ako aj s modifikáciou modelu pre extrémne veľké poistné plnenia s „ťažkým chvostom“. Riziko insolventnosti je tu vyjadrené pomocou pravdepodobnosti ruinovania (krachu). V príspevku prezentujeme niekoľko aproximácií a horných odhadov pre pravdepodobnosť krachu a venujeme sa podmienkám, za ktorých uvažované modely sú korektné.