Selection of optimal investment portfolio based on the model with two measures of risk

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Abstract
One of the most popular methods to choice the optimal investment portfolio is the method value-risk proposed by Markowitz. This model was analysed for different measures of risk such as: variance, semideviation, value-at-risk or conditional value-at-risk. Different measures of risk is focusing on the different properties of distribution of rate of return. For example the variance measures the dispersion of rate of return and the value-at-risk or conditional value-at-risk measure the probable loss. In this article optimisation models with two measures of risk will be analysed. These models will be applied for different risk measures. The optimal portfolio selected on the base of proposed models will be compared according to level of risk and profitability.

Key words

1. Introduction
Optimal investment portfolios are selected according to two parameters: the rate of return and the value of risk. The optimal portfolio must satisfy two conditions: it should be of lowest risk and highest rate of return. For the construction good portfolios is most commonly used models where risk is minimized and rate of return is maximized. In most calculation we apply single objective model where one of these two parameters is optimized and the other is limited.

Other approach in the construction of optimal portfolios is to use more then only these parameters. The aim of this article is analysis of models where optimal portfolios are selected based on the rate of return and two measures of risk. The problem like that was analyzed for different measures of risk. As example we can mention the paper of Wang [7] which considered optimal portfolio according to rate of return, variance and the Value at Risk. Konno and Yamamoto [2] analyzed model where they used the expected rate of return, variance and skewness, and Konno and Yamazaki [1] proposed mean-absolute deviation-skewness optimization model.

In the first part of the article the standard optimal portfolio problem will be described. Some risk measures such as variance, Conditional Value-at-Risk and Gini’s Mean Difference will be presented. In the next part multiobjective problem of selection of optimal portfolio will be described. The last part of this article is a numerical example. In this example all presented models will be analyzed.

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2. Problem of selection optimal investment portfolio

To determine the optimal investment portfolios are used the mean-risk models. Let’s consider a set of n securities. Denote by $R_j$ the rate of return of j security at the end of the investment period. $R_j$ is a random variable because the future price of security is unknown. By the vector $x = (x_1, x_2, ..., x_n)$ we defined the portfolio where $x_j$ expressing the weights defining portfolio $x$. The rate of return of portfolio $x$ is a random variable defined as $R_x = x_1R_1 + x_2R_2 + ... + x_nR_n$. The distribution of this random variable is defined by the function $F(r) = P(R_x \leq r)$ and is dependent on the choice of $x = (x_1, x_2, ..., x_n)$. The components of the vector $x$ (weights) must satisfy two following conditions: all components should sum to 1 ($\sum_{j=1}^{n} x_j = 1$) and all weights should be non-negative ($x_j \geq 0$), which means the short sales is not allowed.

Let’s consider two portfolios $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ with the rates of return $R_x = x_1R_1 + x_2R_2 + ... + x_nR_n$ and $R_y = y_1R_1 + y_2R_2 + ... + y_nR_n$ respectively. To decide which portfolio of $x$ or $y$ is better than we can use the appropriate preference relation. If $\rho(\cdot)$ is a measure of risk and $E(\cdot)$ is the expected rate of return of portfolio (mean) then the portfolio $x$ is preferred to $y$ if and only if $E(R_x) \geq E(R_y)$ and $\rho(R_x) \leq \rho(R_y)$ with at least one strict inequality. To choose an optimal portfolio according to this relation we can use the bi-criteria optimization model where risk is minimized and expected rate of return of portfolio is maximized. In practice usually we use single objective mean-risk model where the value of risk is minimized and the expected rate of return is a constraint:

$$\rho(R_x) \rightarrow \text{min}$$
$$E(R_x) \geq R_0$$
$$\sum_{j=1}^{n} x_j = 1$$
$$x_j \geq 0 \text{ for } j = 1, 2, ..., n$$

$R_0$ denote the required level of rate of return of portfolio and is defined by the investor.

It is also possible to maximize the expected rate of return and the value of the risk imposed restrictions.

In the model, such as the above, many different measures of risk were used. The most important measure is standard deviation or variance defined as $\sigma^2(x) = E[(R_x - E(R_x))^2]$. It is a standard risk measure and it measures the dispersion of rate of return. The mean-variance model (MV model) is as follows:

$$\sigma^2(R_x) \rightarrow \text{min}$$
$$E(R_x) \geq R_0$$
$$\sum_{j=1}^{n} x_j = 1$$
$$x_j \geq 0 \text{ for } j = 1, 2, ..., n$$

The other measure of risk which we can apply in model like this above are the Conditional Value-at-Risk and the Gini’s mean difference. Using the scenario approach we can received
the mean-risk model with these both measures in the linear form. In practice very often we assumed that the rates of return of portfolio are discrete random variables. These variables can be described by the realizations for T periods. For this purpose we can generate scenarios or use the historical data. Let \( p_i \) denote the probability of scenario \( i \) (for \( i=1,2,\ldots, T \)) and \( \sum_{i=1}^{T} p_i = 1 \). Random rates of return can be defined in discrete probability space. Let \( r_{ij} \) is rate of return of security \( j \) in scenario \( i \) (for \( i=1, 2, \ldots, T \) and \( j=1, 2, \ldots, n \)). This random variable \( R_j \) represents rate of return of security \( j \) by finite distribution \( \{r_{1j}, r_{2j}, \ldots, r_{Tj}\} \) with probability \( p_1, p_2, \ldots, p_T \). Random variable \( R_x = \{R_{x1}, R_{x2}, \ldots, R_{xT}\} \) represents rate of return of portfolio \( x = (x_1, x_2, \ldots, x_n) \). \( R_{xi} \) is a rate of return of portfolio in period \( i \) and is defined in the following way: \( R_{xi} = x_1r_{i1} + x_2r_{i2} + \ldots + x_nr_{in} \).

The Conditional Value-at-Risk measures the expected loss corresponding to a number of worst cases, depending on the chosen confidence level \( \alpha \). The Conditional Value-at-Risk for portfolio \( x \) can be defined as [4, 5]

\[
\text{CVaR}_\alpha(R_x) = \min_{v \in \mathbb{R}} F_{\alpha}(x, v)
\]

where \( F_{\alpha}(x, v) \) is finite and continuous function of \( v \) in the following form:

\[
F_{\alpha}(x, v) = v + \frac{1}{T(1-\alpha)} \mathbb{E}[R_x - v]^+ \]

and

\[
[u]^+ = \begin{cases} 
  u & \text{for } u \geq 0 \\
  0 & \text{for } u < 0 
\end{cases}
\]

In the case when \( R_x \) is a discrete random variable, function \( F_{\alpha}(x, v) \) can be rewrite as follows

\[
F_{\alpha}(x, v) = v + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i \left[ -\sum_{j=1}^{n} x_j r_{ij} - v \right]^+ .
\]

By introducing the additional variables \( u_i \) defined by the condition \( u_i + \sum_{j=1}^{n} x_j r_{ij} + v \geq 0 \), we obtain the mean-CVaR model (MC model) in the linear form [3, 5]:

\[
\begin{align*}
&v + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i u_i \rightarrow \min \\
&E(R_x) \geq R_0 \\
&u_i + \sum_{j=1}^{n} x_j r_{ij} + v \geq 0 \quad \text{for } i = 1, 2, \ldots, T \\
&u_i \geq 0 \quad \text{for } i = 1, 2, \ldots, T \\
&\sum_{j=1}^{n} x_j = 1 \\
x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n
\end{align*}
\]

where \( v, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_T \) are decision variables.

Mean-risk model in which risk is expressed by the Gini’s Mean Difference can also be presented in the linear form. The Gini’s Mean Difference (\( \Gamma \)) for the security \( j \) is defined as [3, 8]:
To calculate the Gini’s Mean Difference (GMD) for portfolio $x$ this following formula is used

$$\Gamma = \frac{1}{2} \sum_{i,k=1}^{T} \left| r_{ij} - r_{ik} \right| p_i p_k$$

Let’s introduce the additional variables defined as $d_{ik} \geq x_{ij} - x_{ik}$ (for $i, k = 1, 2, \ldots, T$). For the scenario data the model with the Gini’s Mean Difference (MG model) is following [8]:

$$\Gamma_p = \frac{1}{2} \sum_{i,k=1}^{T} d_{ik} p_i p_k \rightarrow \min$$

$$d_{ik} \geq \sum_{j=1}^{n} x_{ij} r_{ij} - x_{ik} r_{ik} \quad \text{for } i, k = 1, 2, \ldots, T$$

$$E(R_x) \geq R_0$$

$$\sum_{j=1}^{n} x_j = 1$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n$$

$$d_{ik} \geq 0 \quad \text{for } i, k = 1, 2, \ldots, T$$

Using these both models for different value of assumed level of rate of return we can received the effective optimal solution [3, 5, 8].

3. Selection optimal portfolio according two measures of risk

Selection an optimal investment portfolio can be made according to three criteria: the expected rate of return ($E(\cdot)$) and two measures of risk ($\rho_1(\cdot)$, $\rho_2(\cdot)$). Then the preference relation can be as follows: the random variable $R_x$ is preferred to the random variable $R_y$ if and only if $E(R_x) \geq E(R_y)$, $\rho_1(R_x) \leq \rho_1(R_y)$ and $\rho_2(R_x) \leq \rho_2(R_y)$ with at least one strict inequality. For this preference relation we can prove that the un-dominated effective solutions are Pareto effective solutions of multi-objective problem. In the optimization model of this problem the value of the expected rate of return is maximized and both measures of risk ($\rho_1(\cdot)$, $\rho_2(\cdot)$) are minimized.

These multi-objective models can be transformed to a single objective problem. This can be done by using for example method known as „ε-constrain method“ [6]. In this method one objective function should be optimizing and the remaining objective functions should be transformed into constraints.

Now we present single objective models which can be used to solve the optimal portfolio problem with two measures of risk. In the first model the variance and CVaR will be used as a risk measures. In mean-variance-CVaR model the variance will be minimized and two remaining criteria will be transformed into constraints. Using the scenario approach we received the single-objective model with the linear constraints. The mean-variance-CVaR model (MVC model) is following [6]:

$$\Gamma_{mean} = \frac{1}{2} \sum_{i,k=1}^{T} \left| \bar{r}_{ij} - \bar{r}_{ik} \right| p_i p_k$$
Parameter $z_C$ is the value which the Conditional Value-at-Risk should not exceed. The value $z_C$ was fixed according to results from the mean-CVaR model. It means by using the mean-CVaR model for the different assumed level of rate of return of portfolio we received different value of objective function (value of risk). From all these values was selected the minimum ($z_{C_{\text{min}}}$) and the maximum value of risk ($z_{C_{\text{max}}}$). In mean-CVaR model as $z_C$ we assumed some values from the range $[z_{C_{\text{min}}}, z_{C_{\text{max}}}]$.

Roman, Darby-Dowman and Mitra proved that the efficient solution of this problem is also the efficient solution of the multi-objective problem [6].

As a constraint we can also use the Gini’s Mean Difference instead CVaR. As in the above model, the variance will be minimized and the Gini’s Mean Difference and the expected rate of return will be changed to constraints. In the scenario approach we can apply mean-variance-GMD model (MVG model) with linear constraints as follows:

Parameter $z_G$, which limits the value of Gini’s Mean Difference, is fixed as the parameter $z_C$ but in this case we use results from the mean-Gini model.

The optimal portfolio problem with two measures of risk can be also solved by the linear optimization models. We can consider models with the Conditional Value-at-Risk and Gini’s Mean Difference as a risk measures. In one of these models the Conditional Value-at-Risk will be minimized and the Gini’s Mean Difference will be limited. Let denote this linear model as the mean-CVaR-GMD model (MCG model):
In the next model these two measures will be converted to each other. The Gini’s Mean Difference of portfolio will be minimized and the Conditional Value-at-Risk will be limited. The mean-GMD-CVaR model (MGC model) is following:

\[
\begin{align*}
\nu + \frac{1}{T(1-\alpha)} \sum_{i=1}^{T} p_i u_i & \rightarrow \min \\
\sum_{j=1}^{n} x_j r_{ij} + \nu & \geq 0 \text{ for } i = 1, 2, ..., T \\
\frac{1}{2} \sum_{i=1}^{T} d_k p_i p_k & \leq z_G \\
d_k & \geq \sum_{j=1}^{n} x_j r_{ij} - x_j r_{ik} \text{ for } i, k = 1, 2, ..., T \\
E(R_x) & \geq R_0 \\
\sum_{j=1}^{n} x_j & = 1 \\
x_j, d_k, \nu & \geq 0 \text{ for } j = 1, 2, ..., n \text{ i, k = 1, 2, ..., T}
\end{align*}
\]

For all these presented single objective models we can prove that the optimal solution of the single objective problem is also a Pareto optimal solution of the original multi-objective problem.

4. Numerical example

The numerical example concern analysis optimal portfolios selected on the basis of the all models presented above. In example were used the data from Stock Exchange in Warsaw from 1st January 2009 to 31st December 2009. In the last part of research were used data of quotations from January 2010. All calculations were made on the base of the daily rates of return. From the set of all securities quoted in the analyzed period only 50 securities according to the best rate of return were selected. All data were analyzed in the four periods –
quarters. For all quarters we received the similar conclusions so the results will be present for the first and last period (period 1 and period 4 respectively). In models with the Conditional Value-at-Risk was used the confidence level $\alpha=0.95$.

Using presented models we can check how the additional condition to the second measure of risk affect the composition of the optimal portfolios according to the mean-variance, mean-CVaR and mean-GMD model. Obtained optimal portfolios were compared in terms of the risk, the level of diversification and the value of real rates of return.

All the portfolios constructed on the base of the mean-GMD model consist only of two components. This property is connected with the linear form of model. For the same assumed level of rate of return, optimal portfolios according to mean-variance model have a higher level of diversification than the appropriate optimal portfolio according to mean-CVaR model. According to results from the mean-CVaR and mean-GMD models we fixed the range for parameters $z_C$ and $z_G$. For example, for the data from period 4 the value of risk in mean-CVaR model was changing from 0 to 0.056069 so parameter $z_C \in [0.002031; 0.056069]$. Values of risk in mean-GMD model are 0.008458-0.021825.

Restriction on the level of the CVaR introduced in the mean-variance model has some influence on the composition of optimal portfolios. For the same assumed level of rate of return optimal portfolios according to mean-variance-CVaR model have higher or the same level of diversification like appropriate optimal portfolio according to mean-variance. If we compare the results from the mean-variance and mean-variance-GMD model we can notice that the composition of the corresponding optimal portfolios do not differ fundamentally. Results for selected optimal portfolio with the same assumed level of the rate of return portfolio are presented in table 1 and 2. The symbol MVC (008) means the mean-variance-CVaR model where the Conditional Value-at-Risk was limited by $z_C=0.008$.

<table>
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<tr>
<th>security</th>
<th>MV</th>
<th>MVC (008)</th>
<th>MVC (012)</th>
<th>MVC (048)</th>
<th>MVC (031)</th>
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<td>ABPL</td>
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<td>0.0206</td>
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<td>AZOTYTARNOW</td>
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<td>0</td>
<td>0</td>
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<tr>
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<td>0</td>
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<td>0</td>
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<tr>
<td>IMMOEAST</td>
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<td>0.0480</td>
<td>0.0450</td>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>0.0655</td>
<td>0.1953</td>
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<tr>
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<tr>
<td>REMAK</td>
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<td>0.0179</td>
<td>0.0543</td>
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<td>TFONE</td>
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<td>0.0233</td>
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<tr>
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<td>0</td>
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<td>Value of risk</td>
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<td>0.001152</td>
<td>0.000371</td>
<td>0.001146</td>
<td>0.002197</td>
</tr>
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</table>

Table 1: Composition of selected optimal portfolios according to mean-variance, mean-variance-CVaR and mean-variance-GMD model (period 1)
Table 2: Composition of selected optimal portfolios according to mean-variance, mean-variance-CVaR and mean-variance-GMD model (period 4)

In the linear presented models additional condition on the value of the other risk measure change the composition of optimal portfolios. Restriction on the value of CVaR which was introduced in model mean-GMD-CVaR let receiving different optimal portfolios than according mean-GMD model. Moreover optimal portfolios according mean-GMD-CVaR model are more diversified than appropriate optimal portfolios calculated on the base mean-GMD model.

Table 3: Composition of selected optimal portfolios according to mean-CVaR, mean-GMD, mean-CVaR-GMD and mean-GMD-CVaR model (for period 1)
MVC and MVG models. However, we can notice that the higher value of limitation risk (z), the lower value of the objective function we received. All presented models with two measures of risk give the higher value of risk than the models with one risk measure — let’s compare the MG and MGC models or MVC and MVG models. However, we can notice that the higher value of limitation risk (z_G or z_C), the lower value of the objective function we received.
In the last part of research we have assumed that each portfolio will be sold in the consecutive working days starting from 4th January to 29th January 2010. For those days were calculated the rates of return. The obtained data are shown in the table 5.

At first let’s compare the portfolio in two groups: non-linear and linear models. For each day in both group, the highest rate of return was marked in bold. We can notice that the MVC and MVG models give better results than MV model. For the linear models we can not clearly say which of them gives the best rate of return. Comparing results from all models we can only say that the models with two measures of risk most often give better results than the models with only one measure of risk.

Reference


Summary

The aim of this article is analysis of models where optimal portfolios are selected based on the rate of return and two measures of risk. As a measures of risk we considered variance, Conditional Value-at-Risk and Gini’s Mean Difference. To solve this multi-objective problem we can use optimization models with one objective function. Using this method we receive the optimal portfolios which give better results in the future than the optimal portfolios selected on the base of the model with one measure of risk.