Introduction to copula functions and their application in portfolio and risk management

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Abstract
Article deals with modeling of mutual dependencies between financial assets. Its aim is to introduce copula functions, which enable to separate the modeling of dependency features of financial assets from the modeling of marginal distribution characteristics, in context of practical portfolio construction tasks. The article briefly describes basic copula functions characteristics, their functional definitions and properties. The main types of copula functions are introduced together with selected questions of parameter estimation. Practical part solves portfolio selection problem and subsequent analysis of instrument riskiness utilizing copula functions. The difference between portfolios constructed using normal copula and student t copula is showed. As expected, the exclusive use of linear correlation coefficients leads to underestimation of total portfolio risk.

Keywords
copula function, correlation, financial modeling, portfolio management, risk management.

1 Introduction

Financial applications often deal with the multivariate distributions of random numbers. The obvious task for financial modellers is to describe the features and behaviour of these multivariate distributions. Usually the whole multivariate distribution of random numbers is estimated. The easiest possibility is to use the well known Normal distribution. In a Gaussian world, the dependency structure of the multivariate random number is captured by a concept of correlations and depicted by a covariance matrix. There can be found many drawbacks of “normal” specification of the dependency structure in non-Gaussian world. A paper by Embrechts et al. (2002) has identified and illustrated several major problems associated with a correlation coefficient, defined as Pearson’s product moment:

- The correlation coefficient is a measure of linear association of random variables and as so, it cannot capture non-linear dependencies.
- Feasible values for the correlation depend on the marginal distributions. Thanks to the specification of the correlation as a scaled covariance, the correlation coefficient will always be influenced by the distribution of marginals.
- Perfect positive dependence does not imply a correlation of one.
- Zero correlation does not imply independence.

Another problem is that the correlation describes the dependency as one single number rather than expressing it as some functional (e.g. the returns of financial assets usually possess higher dependency in lower tails of their multivariate distribution). Figure 1 graphically depicts the dependency structure of two return series, namely the returns of stock indexes S&P 500 and DJ EuroStoxx 50. It shows a scatter plot of percentiles of corresponding time series. The

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sample correlation coefficient of these indexes when estimated using over 9 years of daily data is 0.459. Figure 2 shows the scatter plot of two normally distributed random numbers, a and b, with the same correlation. The stronger tail dependence (occurrence of returns in the tails of marginal distribution) of real world returns is obvious.

These pitfalls of the correlation coefficient lead to many failures of financial application. In risk management or actuarial finance, the estimates of risk metrics, like Value at Risk, possibly underestimate risks. In the area of portfolio management, the optimal portfolios could be far less diversified than expected. As a result, a new assessment of dependency modelling was proposed in the last decade of 20th century (Embrechts, 2008). Sklar’s theorem is used to show the existence of specific function, called copula, which links together pure univariate features of marginal distributions into the whole multivariate distribution. The copula function enables to independently model the specification of marginal distributions and the purely joint features. In this context, stronger tail dependency can be justified as well as skewed behaviour of random numbers.

The objective of this paper is to introduce the concept of copulas and its usage in practical problems. In the first part we define the copula function together with the most important topics: its representation, elliptical and Archimedean copulas and estimation. In the second part we show the numerical examples which demonstrate the importance of copula modelling in portfolio and risk management. The last part concludes.

2 Copulas

2.1 Definition of copulas

The distribution of a multivariate random variable \( X \) can be factored into two separate components. The marginal distributions of each entry vector \( X \) represent the purely univariate features of \( X \). On the other hand, the purely joint component of the distribution of \( X \) can be summarized in standardized distribution, copula. The copula represents the true interdependence structure of the random variable. Intuitively, the copula is a standardized version of the purely joint features of the multivariate distribution, which is obtained by filtering out all the purely one-dimensional features, namely the marginal distributions of each entry \( X_n \).

\(^2\) It is important to point out that the sample correlation estimated from daily data is exposed to serial correlation. Daily returns in international equity markets are likely to display some form of serial correlation because markets in different countries are open at different times and so the processing of new information is biased.
In order to factor out the marginal components, it is needed to deterministically transform each entry $X_n$ in a new random variable $Un$, whose distribution is the same for each entry. Since all $Un$ have the same distribution, the univariate features of $X$ are removed. It is natural in financial modelling to consider cumulative distribution function $FX$ to map a generic random variable $X$ into a random variable $U$ (following Meucci, 2005, the random variable $U$ is called the grade of $X$) and reads:

$$U = FX(X).$$

The grade of $X$ is a deterministic transformation of the random variable $X$ that assumes values in the interval $[0, 1]$. In particular, each marginal component $X_n$ can be standardized by means of uniform distribution. The random variable $U$ can be expressed as the vector of the grades:

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_N \end{pmatrix} = \begin{pmatrix} FX_1(X_1) \\ \vdots \\ FX_N(X_N) \end{pmatrix}.$$

In other words, the random variable $U$ represents percentiles of random variable $X$. The copula of the multivariate random variable $X$ is the joint distribution of its grades. Sklar’s theorem shows, that given any multivariate random variable $X$ with continuous marginal distributions, there is a unique copula function $C$ such that:

$$FX(x_1, \ldots, x_N) = C(FX_1(x_1), \ldots, FX_N(x_N)).$$

Corresponding probability density function of the multivariate random variable can be expressed as the product of the pdf of copula and the pdf of the marginal densities of its entries:

$$f_X(x_1, \ldots, x_N) = f_C(FX_1(x_1), \ldots, FX_N(x_N)) \prod_{n=1}^N f_{X_n}(x_n).$$

Since the copula is a distribution, namely distribution of the grades $U$ (or percentiles of the random number realization) it can be represented in terms of the probability density function or the cumulative distribution function, or the characteristic function. It is proved in appendix 2.3 of Meucci (2005) that the pdf of the copula reads:

$$f_C(u_1, \ldots, u_N) = \frac{f_X(Q_{X_1}(u_1), \ldots, Q_{X_N}(u_N))}{f_{X_1}(Q_{X_1}(u_1)) \cdots f_{X_N}(Q_{X_N}(u_N))},$$

where $Q_{X_n}$ is the quantile function (or inverse cdf) of generic $n$–th marginal entry of $X$. The copula of random variable $X$ could be equivalently represented in terms of its cumulative distribution function, which reads:

$$F_C(u_1, \ldots, u_N) = FX(Q_{X_1}(u_1), \ldots, Q_{X_N}(u_N)).$$

There are two more concepts worth mentioning, namely tail dependence and bounds for dependence. The lower/upper tail dependence coefficient represents the conditional probability that one random variable takes a value in its lower/upper tail, given that the other random variable takes a value in its lower/upper tail. Copula is said to have symmetric tail dependence if theirs lower and upper tail dependence coefficients equal. Conversely, it has asymmetric tail dependence if its tail dependence coefficients differ. There exist independence copulas, copulas with perfect positive dependence and copulas with perfect negative dependence. Perfect positive/negative dependency is defined as Fréchet upper/lower bound copulas. No other copula can take value that is greater than the value of Fréchet upper bound copula and no other copula can take value that is less than the value of Fréchet lower

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3 For further discussion see Alexander (2008).
bound copula. As it is highlighted in Alexander (2008), less than perfect (positive or negative) dependence is linked to certain parametric copulas. Copula captures positive or negative dependence between the variables if it tends to one of Fréchet bounds as its parameter values change. The Gaussian copula does not tend to Fréchet upper bound as the correlation increases to 1 and neither does it tend to Fréchet lower bound as approaching correlation of -1.

2.2 Different types of copulas

There exist many methods to derive copula functions (Nelsen, 2006). The most used are *inversion method* and *generation functions method*. The inversion method derives copula representations from multivariate distributions such as normal or student t distribution. The most common examples of inversion method derived copulas are elliptical copulas. An alternative method for building copulas is based on a generator function. These copulas are called Archimedean copulas.

2.2.1 Elliptical copulas

A Gaussian copula is derived using inversion method from the multivariate and univariate standard normal distribution functions, denoted Φ and φ. It is defined as:

\[ C(u_1, \ldots, u_n; \Sigma) = \Phi(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_n)) \]

The cumulative distribution function of normal copula cannot be written in a simple closed form. The normal copula probability density function is given by:

\[ c(u_1, \ldots, u_n; \Sigma) = |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \xi'(\Sigma^{-1} - I)\xi\right) \]

where Σ denotes the correlation matrix, |Σ| is its determinant and \( \xi = (\xi_1, \ldots, \xi_n)' \) where \( \xi_i \) is the \( u_i \) quantile of the standard normal random variable \( X_i \). The only one unknown parameter is, in this case, the correlation matrix Σ. The normal copula density is calculated as follows:

1. Firstly, the grades of marginals are quantified using cumulative distribution functions of corresponding marginals: \( u_i = F_{X_i}(x_i) \) for \( i = 1, \ldots, n \). It is important to highlight, that copula modelling enable to use different probability specification for every marginal. Also empiric cdf could be used.
2. Apply the quantile normal function: \( \xi_i = \phi^{-1}(u_i) \) for \( i = 1, \ldots, n \).
3. Use the correlation matrix Σ to calculate the normal copula density.

Opposite sequence can be used when simulating from copula distribution. Figure 3 shows the probability density function of a bivariate normal copula with correlation coefficient 0.5 as a function of \( u_1 \) and \( u_2 \) which each range from 0 to 1. The normal copulas have symmetric tail dependence behaviour, what is also obvious from graphical representation. They have zero or very weak tail dependence unless the correlation is 1. When also the marginals are normally distributed, there is always zero tail dependency. Obviously, this is not appropriate for modelling dependencies between financial assets.

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4 \( u_i = P (X_i < \xi_i), X_i \sim N (0, I), i = 1, \ldots, n. \)
Another elliptical copula, which is derived implicitly from a multivariate distribution function, is Student t copula. It is defined by:

\[ C_v(u_1, \ldots, u_n; \Sigma) = t_v\left( t_v^{-1}(u_1), \ldots, t_v^{-1}(u_n) \right) \]

\( t_v \) and \( t_v \) are multivariate and univariate Student t distribution functions with \( \nu \) degrees of freedom and \( \Sigma \) denotes the correlation matrix. Like the normal copula, the Student t copula cumulative distribution cannot be written in a single closed form. The Student t copula probability density function is defined as:

\[ c_v(u_1, \ldots, u_n; \Sigma) = K \prod_{i=1}^{n} \left( 1 + \nu^{-1} \xi_i \right)^{(\nu+1)/2} \prod_{i=1}^{n} \left( 1 + \nu^{-1} \xi_i^2 \right)^{\nu/2}, \]

where \( \xi = \left( t_v^{-1}(u_1), \ldots, t_v^{-1}(u_n) \right) \) is a vector of realization of Student t variables and \( K() \) is a gamma function defined as:

\[ K = \Gamma\left( \frac{\nu}{2} \right) \Gamma\left( \frac{\nu + 1}{2} \right)\Gamma\left( \frac{\nu + n}{2} \right). \]

Figure 4 shows a bivariate t copula probability density function with 4 degrees of freedom and correlation of 0.5 also as a function of \( u_1 \) and \( u_2 \). Note that the peaks in the tails are symmetric because the copula has symmetric tail dependency but they are higher than those of normal copula with correlation of 0.5 because the t copula has stronger tail dependence. However, a combination of the normal copula and student t marginals can “create” multivariate distribution with stronger tail dependence. Although both normal and student t copulas are elliptical copulas with symmetric distributions\(^5\), for practical purposes, a shape of whole distribution of random variable \( X \) (the joint distribution of marginals and copula) is important. The combinations of elliptical copulas and asymmetrically distributed marginals give raise to non symmetrical behavior of random variables, which are also important in financial modeling, especially when derivatives are considered.

2.2.2 Archimedean copulas

Archimedean copulas are constructed using generator functions method. Given any generator function \( \Psi \), it is possible to define the corresponding Archimedean copula as:

\(^5\) There exist wide variety of Student t copulas, many of them with asymmetric tail dependence (see e.g. Demarta and McNeil, 2005).
\[ C_A(u_1, \ldots, u_n) = \Psi^{-1}(\psi(u_1) + \ldots + \psi(u_n)). \]

Its associated density function is
\[ c_A(u_1, \ldots, u_n) = \Psi_{(n)}^{-1}(\psi(u_1) + \ldots + \psi(u_n)) \prod_{i=1}^{n} \Psi'(u_i), \]
where \( \Psi_{(n)}^{-1} \) is the \( n \) – th derivative of the inverse generator function. When the generator function \( \Psi(u) = -\ln(u) \) the Archimedean copula becomes the independent copula. More generally, the generator function can be any strictly convex and monotonic decreasing function. Hence, there exists a large number of different Archimedean copulas. Alexander (2008) introduced two simple Archimedean copulas that are used in market risk analysis thanks to its asymmetric tail dependence. These are Clayton and Gumbel copulas. The corresponding generator function of Clayton copula is defined as
\[ \Psi_{\text{Clayton}}(u) = \alpha^{-1}(u^{-\alpha} - 1), \alpha \neq 0, \]
and the generator function of Gumbel copula as
\[ \Psi_{\text{Gumbel}}(u) = -(\ln(u))^{\delta}, \delta \geq 1. \]

Interested reader is referred to Alexander (2008) or Nelsen (2006) for further discussion of Archimedean copulas.

### 2.3 Estimation of copulas

The multivariate distribution of the random number \( X \) comprises from the univariate distributions of its marginals and the distribution of the copula. In general it is possible to estimate the copula and marginals parameters together in one step. However this “aggregated” estimation approach can become too complex to formulate and to effectively solve. When estimating the copula is the primary objective, the unknown marginal distributions of the data enter the problem as unnecessary parameters. The first step is usually to quantify the grades of the marginals using its cumulative distribution functions. In general, the marginal modelling can be done by means of (i) fitting parametric distribution to each marginal, (ii) modelling the marginals nonparametrically using a version of the empirical distribution functions, or (iii) using a hybrid of the parametric and nonparametric methods. After specifying the grades, the copula parameters can be estimated either by the full Maximum Likelihood Estimation or in some cases calibrated by the Generalized Method of Moments (Demarta and McNeil, 2005). When the copula probability density function is known, constructing the log likelihood function is straightforward. In the full Maximum Likelihood Estimation (MLE Thereafter), all unknown parameters are estimated in one step. Using the full MLE method is fairly robust when the number of dimensions is low, but optimization of the likelihood function can become cumbersome when the quantity of estimated parameters increases (this can be seen when dealing with the elliptical copulas where the number of unknown parameters in the correlation matrix can grow quickly). It is also possible to use the Generalized Method of Moments when estimating special cases of copulas, e.g. student t copulas. The Method of Moments exploits the correspondence between copulas and rank correlations. For example, it can be shown that Kendall’s tau \( \tau \), has a direct relationship with a bivariate copula function \( C(u_1, u_2) \) as follows:
\[ \tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \]

If the copula depends on one parameter then it can be calibrated using a sample estimate of the rank correlation. For instance, the bivariate normal copula depends on one parameter, the correlation coefficient \( \rho \), and the above relation yields:

\[ \rho = \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \]

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\[^6\] For detailed discussion about rank correlations see Alexander (2008).
This relationship between the sample estimate of Kendall’s tau and correlation coefficient holds also for other elliptical copulas, like student t copula. There can be found also relationships between rank correlations and non elliptical copulas in the literature\(^7\). After specifying the correlation coefficient using the rank correlation, the rest of unknown parameters (e.g. degrees of freedom) are estimated using MLE. The Method of Moments is suitable when dealing with higher dimensionality. The all components of the correlation matrix can be thus estimated using the sample Kendall’s tau. As in every MLE procedure the best fit from parametric specifications can be determined by Akaike information criterion or Bayesian information criterion.

3 Numerical examples

In this part, we present numerical examples to highlight the importance of copula modelling when constructing portfolios and when assessing the riskiness of individual positions. The crucial role of copulas in modelling the behaviour of portfolio, which invests in derivative instruments, is clear. The pay off of derivatives is usually non-linear and therefore the use of copulas is natural. When the portfolio invests only in stocks and bonds, the need for copula formulation of dependence structure is not obvious.

We assume long only mutual or pension fund which invests in 3 bond market indexes and 5 stock indexes. Considered bond investments are represented by iBoxx\(^8\) indexes and cover US Treasuries, Euro Government Bond Market and Euro A Corporate Bond Market. Considered stock investments are: S&P 500, DJ EuroStoxx 50, CECE Traded Eur Index, FTSE 100 and Nikkei 225. All calculations are done using daily data from 1.1.1999 to 24.6.2008, what sums to 2472 daily observations of return data.

Two copula specifications are compared, namely normal copula and student t copula. The underlying copula parameters were estimated using full MLE and optimal portfolios were calculated by means of mean–CVaR optimization. The riskiness of individual positions was formulated as marginal contribution to CVaR.

3.1 Portfolio management

This section compares the differences between optimal portfolios that are estimated using normal and student t copulas. In both cases, we isolated marginal distributions using empirical cumulative distribution functions \(P^E(x)\). Thus the skewness and/or the kurtosis of marginal distributions are preserved. The parameters to be estimated are the correlation matrix in the case of normal copula and the correlation matrix and the degrees of freedom in the case of student t copula. The estimation of normal copula correlation matrix is straightforward, as its MLE is sample estimator. Recall that the copula operates on grades or percentiles of underlying distributions. The estimation of copula and simulation from its distribution can be visualized as:

\[
\text{return data} \xrightarrow{P^E(x)} \text{grades} \xrightarrow{\varphi^E} \text{copula data} \xrightarrow{} \text{correlation matrix estimate} \\
\text{return data} \xleftarrow{P^{E^{-1}}} \text{grades} \xleftarrow{\varphi^{E^{-1}}} \text{simulation from copula}
\]

The parameters of student t copula were estimated using full MLE method as in Meucci (2006). The estimated number of degrees of freedom was 12. The values of log likelihood

\[^7\text{See Nelsen (2006) or Alexander (2008).}\]

\[^8\text{www.indexco.com or www.markit.com}\]
function were not elastic to changes to the number of degrees of freedom after reaching the level of 12. The value of log likelihood function grew by 0.5% when the number of degrees of freedom was increased by one.

After specifying the parameters of normal and student t copula we generated 10 000 Monte Carlo simulations from corresponding copula distribution using algorithms proposed in Alexander (2008). To “come back” to return distributions we used inverse empirical cumulative distribution. We used equilibrium – like assumption concerning the mean value of each asset’s return (Black and Litterman, 1990). The mean of each return distribution was scaled to be proportional to two times its 5% historical one day VaR. In practice, scaling of mean values can be done with Black – Litterman model of expected returns. The optimal portfolios were estimated by means of mean – Conditional Value at Risk optimization, following Rockafellar and Uryasev (2000). Standard full investment and long only constrains were imposed, the required confidence level were set to 5% in one day investment horizon. In mean – CVaR optimization, portfolio manager maximizes expected return for given level of conditional value at risk. Table 1 shows optimal portfolios created using normal and student t specification of copula function. It summarizes their compositions, expected values, standard deviations and tail risk statistics:

<table>
<thead>
<tr>
<th></th>
<th>normal</th>
<th>student t</th>
<th>normal</th>
<th>student t</th>
<th>normal</th>
<th>student t</th>
</tr>
</thead>
<tbody>
<tr>
<td>exp.mean</td>
<td>1.29%</td>
<td>1.28%</td>
<td>2.77%</td>
<td>2.73%</td>
<td>3.99%</td>
<td>3.93%</td>
</tr>
<tr>
<td>std (p.a.)</td>
<td>2.94%</td>
<td>2.93%</td>
<td>7.51%</td>
<td>7.24%</td>
<td>12.30%</td>
<td>12.24%</td>
</tr>
<tr>
<td>VaR</td>
<td>-0.31%</td>
<td>-0.31%</td>
<td>-0.75%</td>
<td>-0.75%</td>
<td>-1.28%</td>
<td>-1.24%</td>
</tr>
<tr>
<td>CVaR</td>
<td>-0.40%</td>
<td>-0.40%</td>
<td>-1.00%</td>
<td>-1.00%</td>
<td>-1.70%</td>
<td>-1.70%</td>
</tr>
<tr>
<td>Euro Goverment Bonds</td>
<td>33.24%</td>
<td>40.77%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>US Treasuries</td>
<td>24.46%</td>
<td>22.28%</td>
<td>44.48%</td>
<td>47.08%</td>
<td>8.00%</td>
<td>12.74%</td>
</tr>
<tr>
<td>Euro Corporates</td>
<td>25.62%</td>
<td>21.47%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>5.72%</td>
<td>4.17%</td>
<td>20.71%</td>
<td>17.79%</td>
<td>32.77%</td>
<td>28.82%</td>
</tr>
<tr>
<td>DJ EuroStoxx</td>
<td>0.83%</td>
<td>1.79%</td>
<td>2.94%</td>
<td>3.19%</td>
<td>5.55%</td>
<td>6.32%</td>
</tr>
<tr>
<td>CECE Index</td>
<td>2.35%</td>
<td>2.76%</td>
<td>11.21%</td>
<td>11.28%</td>
<td>20.02%</td>
<td>20.37%</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>3.44%</td>
<td>2.82%</td>
<td>3.78%</td>
<td>4.08%</td>
<td>3.70%</td>
<td>2.47%</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>4.35%</td>
<td>3.95%</td>
<td>16.87%</td>
<td>16.59%</td>
<td>29.16%</td>
<td>29.28%</td>
</tr>
<tr>
<td>% of stock investments</td>
<td>16.68%</td>
<td>15.49%</td>
<td>55.52%</td>
<td>52.92%</td>
<td>91.20%</td>
<td>87.26%</td>
</tr>
</tbody>
</table>

Table 1: Comparison of optimal portfolios

As can be seen from Table 1, the normal copula leads to under estimation of total portfolio risk, when it puts more weight to more risky instruments. The stock investments often suffer from heavy tails and strong lower tail dependence. The student t copula, “only” with 12 degrees of freedom, was able to capture these empirical observations. Also, the student t copula creates more diversified portfolios, which propose diversification benefits in negative market developments as well. However, the representation of Japan is striking. Thanks to different trading time the new information are processed with time lags in some markets. When using daily data it causes a serial correlation. So the serial correlation not adjusted estimator can lead to spurious results.

3.2 Risk Management

The common task for risk managers is to express the riskiness of individual positions held in portfolios. The concept of marginal and absolute contributions to risk (Litterman, 1996) can be used to quantify these characteristics. Marginal contribution to risk (MCTR thereafter) of asset $i$ indicates by how much the risk metric changes if the weight of asset $i$ is increased by 1% (funded by cash). Absolute contribution to risk (ACTR thereafter) translates the MCTR of asset $i$ into percentage expression of total risk. The concept of marginal and absolute risk contributions was extended to different risk measures and different areas of use (see e.g. Meucci, 2007 or Mausser, 2001). In our example we expressed the differences between MCTRs and ACTRs, which were estimated for the hypothetical portfolio using normal and
student t copula. The risk metric employed was, as in preceding example, the Conditional Value at Risk. The MCTRs and ACTRs were estimated by upper empirical cumulative distribution function method from simulated data, as presented in Mausser (2001). Table 2 shows weights of the hypothetical portfolio, prefix $n$ indicates normal copula specification, whereas prefix $s$ indicates student t copula specification.

<table>
<thead>
<tr>
<th>% weights</th>
<th>n MCTR</th>
<th>n ACTR</th>
<th>s MCTR</th>
<th>s ACTR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euro Government Bonds</td>
<td>45%</td>
<td>-0,07%</td>
<td>13,1%</td>
<td>-0,06%</td>
</tr>
<tr>
<td>US Treasuries</td>
<td>25%</td>
<td>-0,05%</td>
<td>9,5%</td>
<td>-0,04%</td>
</tr>
<tr>
<td>Euro Corporates</td>
<td>5%</td>
<td>-0,01%</td>
<td>1,4%</td>
<td>-0,01%</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>9%</td>
<td>-0,13%</td>
<td>25,2%</td>
<td>-0,14%</td>
</tr>
<tr>
<td>DJ EuroStoxx</td>
<td>5%</td>
<td>-0,10%</td>
<td>20,6%</td>
<td>-0,10%</td>
</tr>
<tr>
<td>CECE Index</td>
<td>4%</td>
<td>-0,07%</td>
<td>13,4%</td>
<td>-0,07%</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>3%</td>
<td>-0,05%</td>
<td>9,3%</td>
<td>-0,05%</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>4%</td>
<td>-0,04%</td>
<td>7,5%</td>
<td>-0,04%</td>
</tr>
<tr>
<td>100%</td>
<td>-0,499%</td>
<td>100,0%</td>
<td>-0,511%</td>
<td>100,0%</td>
</tr>
</tbody>
</table>

*Table 2: Comparison of risk statistics*

The under estimation of tail risk when normal copula specification is used is evident. The estimate of “normal copula” CVaR on 5% confidence level and one day investment horizon is a loss of 0.499% of portfolio value. The corresponding estimate of “student t copula” CVaR is the loss of 0.511%. The different MCTRs show only minor differences. However, the corresponding ACTRs lead to different perception of risk. When the normal copula is applied, the percentage of total risk stemming from stock positions is 76%, whereas under student t copula, it is 79.2%. Applying the different copulas can result in different decomposition of risk and hence to possible faults of risk management applications.

4 Conclusion

The copula functions can help to realistically describe the true dependencies between random numbers. There exist many different classes of copulas. For modelling fat tailed dependencies, the elliptical class of copulas is usually sufficient. When the asymmetric dependencies are present, the Archimedean copulas are better choice. Moreover, the copula modelling allows using different approaches for modelling the marginal distributions and the interdependence structure. The potential problem is the estimation of parametric specification of copula when the number of dimensions is large. Simulation algorithms are another task for further research. The simulation from Archimedean copulas encompasses working with quantile copula functions which can become cumbersome. The simplified examples showed better properties of student t copula then normal in the field of portfolio management and risk analysis. They introduced also the problem of serial correlation. When dealing with daily data from international markets it is necessary to add serial correlation adjustments to copula modelling framework.

References

