The convergence of binomial and trinomial option pricing models

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Abstract

We can get the fair value of any financial derivative either analytically, by simulation or applying suitable numerical technique. For example, when we solve the American option valuation problem, the lattice models can be very useful. The procedure is based on modeling of the underlying asset price evolution (in discrete time with discrete price increments) and subsequent evaluation of the payoff function for all scenarios. Next, the initial fair value is obtained by backward procedure. The evolution of the underlying asset can be modeled e.g. by binomial model, trinomial model, etc. In this paper we study the principles of binomial and trinomial models and, since we can treat them as an approximation of the continuous time model, their convergence. More particularly, we examine the convergence of European call and American put options. The application possibilities of lattice models are connected to the firm management through the broad field of real options.

Keywords

Option pricing, American option, binomial model, trinomial model, convergence

1. Introduction

Financial derivatives play a crucial role in financial decision-making and risk management. It is natural that huge attention is devoted to various replication, pricing and hedging methods. The most popular present-time models and procedures arose in 70’s of the last century and were built up on the famous Black and Scholes approach [2].

The original Black and Scholes model was intended to price a European call option on no-dividend paying stock with normally distributed returns. Subsequent research results into more advanced models incorporating, besides others, underlying assets with regular payment either in the form of dividend, domestic and foreign riskless rate, convenience yield, including storage costs (see e.g. Hull [11] for details on particular models), non-Gaussian character of returns, either through jump models or modeling the skewness and kurtosis (see e.g. Cont and Tankov [7] for impressive review), more complicated payoff functions, stochastic parameters (volatility, riskless rate, internal time) and incomplete markets (transaction costs, constrains on portfolio positions, etc.).

Moreover, there were developed new techniques or methodologies, regularly applied in other fields of science, were adapted to finance applications, such as Monte Carlo

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simulation (Boyle [3], Boyle et al. [5] and Glasserman [10]) or Finite Difference Method (Topper [15] or Zhu et al. [16]).

Very popular approach, which allows us to valuate also American options is generally referred to as the lattice method. In fact, it was firstly applied to option pricing by Cox et al. [8] as an approximation to (at that time) too complicated Black and Scholes model. To be more specific, it was supposed that the underlying asset price can move either down or up within particular time interval (hence, the model is also called binomial). Later, the procedure was extended to allow the price movements to have also third state, hence, the trinomial model (see Boyle [4]). Subsequently, there were developed many other procedures, such as models for multiple sources of risk or models with variable time period.

Besides that, the option approach has started to be applied in decision-making and financial management of companies. This special field of application belongs to so called real option problem. Since real options usually has the property of complex American (or Bermudan, i.e. discrete American) options, the application strength of lattice models increases.

In this paper, we aim at the two most common types of lattice models – the binomial and trinomial. The task of the paper is to study the convergence of these two models for selected pricing problems – the European call and American put option valuation.

We proceed as follows. Section 2 is devoted to option valuation problem in general. In subsequent sections, we define and derive binomial and trinomial model of option pricing. Finally, in Section 4 we provide several application examples.

2. Option pricing problem

An option is a very interesting type of financial derivative. A typical characteristic is that the payoff function is of the non-linear form. It implies broad spectrum of applications within the financial risk management process for both, financial as well as non-financial subjects. Simultaneously, it is the reason why the option valuation procedure can be more or less complicated.

In general, we can define the option as a financial security, which gives its owner the right to execute particular trade with the underlying asset, we denote it by $S$, at maturity time, $T$, under predefined conditions. The right to make a trade can be very broad. Usually, it is either the right to buy (a call option) or the right to sell (a put option). In these basic cases, the owner has the right to buy (sell) the underlying asset for the fixed and predetermined cash amount, called the strike price, $K$. Alternatively, the underlying asset can be changed for another asset.

Moreover, there are several other contract conditions providing more details about option exercising. For example, the strike price or the ability of exercising can depend on the path followed by the underlying asset price. If there are not any additional contract specification, we can denote the option as the plain vanilla. In other cases, the option is referred to as the exotic option.

The cash flowing from the option at maturity is specified by the payoff function. Now, we define the payoff function of vanilla call $\Psi_{\text{call}}^{\text{vanilla}}$:

$$\Psi_{\text{call}}^{\text{vanilla}} = (S_T - K)^+$$

and vanilla put $\Psi_{\text{put}}^{\text{vanilla}}$:

$$\Psi_{\text{put}}^{\text{vanilla}} = (K - S_T)^+,$$
where \((x)^+ \equiv \max(x, 0)\).

Furthermore, the option, which can be exercised only at maturity time \(T\) is called a European. By contrast, an American option is the one, which can be exercised at any time during the option life, \(\tau = T - t\).

The option valuation problem is usually solved within the risk-neutral setting. The justification is that being able to replicate the financial derivative or hedge its payoff perfectly, the type of the world (risk neutral or statistically true) within which we solve the valuation problem should not change the result. From the mathematical point of view, we must be able to find unique equivalent martingale measure \((the \ set \ of \ probabilities \ Q)\) under which the relevant underlying process will behave as a martingale.

In general, we can get the initial option value \(V_0\) by discounting the payoff expected at maturity. It is useful to take the expectation within the risk-neutral world and discount the result by the riskless rate \((r)\):

\[
V_0 = e^{-r\tau} E^Q_{0, T} [\Psi_T]. \tag{3}
\]

To be strictly, this formulation is not valid for American options, since it can be exercised at any time \(T \in [0, T]\). Thus, we can reformulate equation (3) in this way:

\[
V_0 = \sup_T e^{-rT} E^Q_{0, T} [\Psi_T], \text{ where } T \in [0, T]. \tag{4}
\]

3. Binomial model

Under classical single-period binomial model (see Cox et al. [8]) it is supposed that knowing the present price the price of any risky asset can take two values in the next time moment. Consider one risky asset, say stock \(S(t)\), with price at time zero \(S_0\), and one riskless asset \(B(t)\), which gains riskless rate \(r\), i.e. \(B(1) = B(0)(1 + r)\).

Under simple binomial model we suppose, that there is one source of uncertainty, say \(Z\), which value at time one can be described by

\[
Z = \begin{cases} 
  u & \text{with probability } p \\
  d & \text{with probability } 1 - p.
\end{cases} \tag{5}
\]

It implies that the stock price at time one can be written as

\[
S_1 = S_1(Z) = \begin{cases} 
  S_0u & \text{with probability } p \\
  S_0d & \text{with probability } 1 - p.
\end{cases} \tag{6}
\]

The parameters \(u\) and \(d\) in equation (6) can be interpreted as indices of up and down movements in the price, respectively. Alternatively, we can formulate the (discrete-time) returns \(\mu\) of the asset price conditionally on \(Z\) as

\[
\mu(Z) = \begin{cases} 
  1 - u & \text{with probability } p \\
  1 - d & \text{with probability } 1 - p.
\end{cases} \tag{7}
\]

Here, \(p \geq 0\) is the true market probability from a set of such probabilities \(P\). Note, that if \(u\) is the index of an up movement, it is higher than \(d\) and to the model make sense, the
riskless return (index of riskless change \( R = 1 + r \) to be more exact) must lie between \( u \) and \( d \) indices. Hence, the basic market condition is

\[
d \leq 1 + r \leq u.
\]  

(8)

Suppose for a moment that (8) does not hold – for example, \( 1 + r \geq u \). This means that whichever the probabilities of up and down movements are, the return of the risky asset is no longer higher than the return of the riskless asset. This implies that under standard assumption of risk aversion, no one will intend to invest into the risky asset.

The standard approach to price any derivative asset \( f \) is based on the no-arbitrage condition. Hence, we are trying to construct the replication portfolio \( \mathcal{H} \) which will replicate the value of \( f \) exactly (or perfectly) for all states of the world. For the model (5), the following equality must hold with probability one:

\[
P[f_1(Z) = H_1(Z)] = 1.
\]  

(9)

Thus, the value of the replicating portfolio \( \mathcal{H} \) must be equal to the value of \( f \) independently of \( Z \) (whichever it will be). Hence, the portfolio \( \Pi \), consisting of long position in \( f \) and short position in \( \mathcal{H} \) (or vice versa), will have a deterministic value at time one:

\[
t = 1 : f_1 - H_1 = 0.
\]  

(10)

Since the portfolio is riskless it must earn riskless return \( r \). Clearly, present (or future) value of zero must be always zero. Thus, it also holds that

\[
t = 0 : f_0 - H_0 = 0.
\]  

(11)

Consider now the European option \( f \) with payoff at maturity given by \( \Psi(Z) \). Thus, we have a model with one source of uncertainty \( (Z) \) and two possible states in the future at one side and \( n + 1 \) independent assets (i.e. \( n \) (\( n = 1 \)) independent risky asset \( S \) + one riskless \( B \)) on the other side. This indicates, that the market is complete, we can find the unique risk-neutral probabilities \( Q \) to get the risk-neutral price of the option \( f \) by appraising the unique replicating portfolio \( \mathcal{H} \).

Denote the structure of the replicating portfolio by \( \mathcal{H}(x, y) \), where \( x \) indicates the amount invested into \( B \) and \( y \) into \( S \), both at time zero. Hence

\[
t = 0 : \mathcal{H}(x, y) = xB + yS_0
\]  

(12)

and

\[
t = 1 : \begin{cases}
Z(1) = u & \Rightarrow \mathcal{H}(x, y) = xB(1 + r) + yS_0u \\
Z(1) = d & \Rightarrow \mathcal{H}(x, y) = xB(1 + r) + yS_0d.
\end{cases}
\]  

(13)

We have stated above, that we should be looking for such \( \mathcal{H} \) that its time one value will be equal to the option payoff \( \Psi_T(Z) \) regardless the state \( Z \). Therefore,

\[
t = 1 : \begin{cases}
Z(1) = u & \Rightarrow \Psi(u) = xB(1 + r) + yS_0u \\
Z(1) = d & \Rightarrow \Psi(d) = xB(1 + r) + yS_0d.
\end{cases}
\]  

(14)

Note, that the maturity time is the only moment when we can uniquely determine the financial option value respecting its payoff, \( f_T(Z) = \Psi_T(Z) \), without considering any other conditions. It means that (14) results into two equations with two unknowns, \( x \) and \( y \). Setting \( B = 1 \) and solving gets:

\[
x = \frac{\Psi(d)u - \Psi(u)d}{(1 + r)(u - d)},
\]  

(15)
The no-arbitrage condition should imply that if (14) holds then from (12):

\[ t = 0 : f_0 = xB + yS_0. \]  

(17)

Thus, putting \( x \) and \( y \) from (15) and (16) into (17) we get

\[ f_0 = \frac{1}{1 + r} \left[ q\Psi(u) + (1 - q)\Psi(d) \right]. \]  

(18)

Here,

\[ q = \frac{(1 + r) - d}{u - d} \]  

(19)

can be interpreted as the risk-neutral probability of going up \((u)\) and \((1 - q)\) as the risk-neutral probability of going down \((d)\). Thus the risk-neutral probability space is given by

\[ \mathbb{Q} = \{ P[Z = u] = q, P[Z = d] = 1 - q \}. \]  

(20)

The extension of the single-period binomial model into the \( n \)-period model is straightforward. The risky asset price evolves according to (6) rewritten into \( n \)-period model

\[ S_n = S_0 \prod_k Z_k. \]  

(21)

Similarly, the riskless asset evolution is given by \( B(n) = B(0)(1 + r)^n \).

Knowing the solution of (12) and (13) and applying the backward recursive procedure, we are still able to recover the option value at time \( t \) on the basis of time \( t + 1 \) values. Thus, (18) changes into

\[ f_t(S_t) = \frac{1}{1 + r} \left[ qf_{t+1}(Su) + (1 - q)f_{t+1}(Sd) \right]. \]  

(22)

Taking these results into account, we can formulate a time zero value of an option with general (European) payoff \( \Psi(S_T) \) as

\[ f_0 = \frac{1}{(1 + r)^n} \sum_{j=0}^{n} Co(j^n)q^j(1 - q)^{n-j}\Psi(Su^j d^{n-j}). \]  

(23)

The whole procedure is depicted at Figure 3.

Alternatively, respecting the risk-neutral world, we can make the average value of \( Z \) to be riskless, thus

\[ (1 - q)d + qu = 1 + r. \]  

(24)

Now, we can directly write the option value at the beginning as its future expectation discounted by the riskless rate. Since \( q \) and \((1 - q)\) are the probabilities of moving up and down, respectively, we get:

\[ f_t = \frac{1}{1 + r} \left[ qf_{t+1}(u) + (1 - q)f_{t+1}(d) \right], \]  

(25)

which is equivalent to equation (18) (or 22).
In order to get the fair price of an American option, we must respect the fact, that opposite to European options, the American can be exercised at any moment prior the maturity time $T$. Hence, it is sufficient to modify equation (22) into the following form:

$$f_t(S_t) = \max \left\{ \Psi(S_t), \frac{1}{1+r} \left[ q f_{t+1}(S_t u) + (1 - q) f_{t+1}(S_t d) \right] \right\}. \quad (26)$$

Very important question, when building up the model, is how to set the parameters to fit the resulting distribution of the underlying asset either to real market returns ($\mu$) or to risk-neutral returns ($r$)\(^2\) at one side, and to the variance at the other side. Following Table 1 provides several basic approaches to set the parameters of the binomial process.

Table 1: Selected approaches to calibrate binomial model within risk-neutral world

<table>
<thead>
<tr>
<th>model</th>
<th>parameter $u$</th>
<th>parameter $d$</th>
<th>$p(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRR1 (1979)</td>
<td>$e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t}}$</td>
<td>$e^{-(r-\frac{1}{2}\sigma^2)\Delta t - \sigma \sqrt{\Delta t}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>CRR2 (1979)</td>
<td>$e^{\sigma \sqrt{\Delta t}}$</td>
<td>$e^{-\sigma \sqrt{\Delta t}}$</td>
<td></td>
</tr>
<tr>
<td>JR (1982)</td>
<td>$e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t}}$</td>
<td>$e^{-(r-\frac{1}{2}\sigma^2)\Delta t - \sigma \sqrt{\Delta t}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Amin (1991)</td>
<td>$e^{r \ln[cosh(\sigma \sqrt{\Delta t})] + \sigma \sqrt{\Delta t}}$</td>
<td>$e^{r \ln[cosh(\sigma \sqrt{\Delta t})] - \sigma \sqrt{\Delta t}}$</td>
<td></td>
</tr>
</tbody>
</table>

4. Trinomial model

Although the trinomial model allows the modeled asset to get three distinct states at the end of the interval, still, only one independent risky asset (underlying asset $S$), one independent riskless asset $B$ and one financial derivative $f$ are available. It means, that the market is not complete. Moreover, the set of risk-neutral probabilities $Q$ probably exists, however, it is not unique. Simultaneously, it is not possible to set up unique replication portfolio $H$. Hence, in order to price a financial derivative, it is inevitable to apply the risk-neutral approach.

In other words, and similarly to (22), it must holds, that:

$$f_t(S_t) = \frac{1}{1+r} \mathbb{E}^Q[F_{t+1}] = \frac{1}{1+r} \left[ q_u f_{t+1}(S_t u) + q_m f_{t+1}(S_t m) + q_d f_{t+1}(S_t d) \right]. \quad (27)$$

In general, we can work with six unknown variables – $q_u, q_m, q_d, u, m$ and $d$. Within the most simple case, the additional third state (middle one) is set to be zero increment in price – i.e. $S_{m+\Delta t} = S_t$ and $m = 0$. Next, if we set $d = 1/u$ and $q_m = 2/3$,\(^3\) three unknown variables remain, $q_u, q_d$ and $u$. Note, that we have exactly three equations (expected return, variance and the sum of probabilities $q_u + q_m + q_d = 1$). The latter can be reformulated into:

$$q_d = 1 - 2/3 - q_u = 1/3 - q.$$  

\(^2\)The expected returns of the underlying asset should be identical with the riskless rate when pricing the option.

\(^3\)Thus, we choose one admissible $Q$.  

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After that, we get ($q$ denotes the risk-neutral probability of price increment):

$$E[S_{t+\Delta t}] = S_t e^{r\Delta t} = q S_t u \Delta t + 2/3 S_t + (1/3 - q) S_t d \Delta t$$

and

$$\text{var}\left[\frac{dS(\Delta t)}{S_t}\right] = \sigma^2 \Delta t = q u^2 + 2/3 + (1/3 - q) d^2 - [qu + 2/3 + (1/3 - q)d]^2.$$ 

Hence, we can relatively easy express the system of input parameters as follows:

$$u = \exp(\sigma \sqrt{3\Delta t}), \quad d = 1/u, \quad m = 1, \quad (28)$$

$$q_u = \sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6}, \quad (29)$$

$$q_m = 2/3, \quad (30)$$

$$q_d = -\sqrt{\frac{\Delta t}{12\sigma^2}} \left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6}. \quad (31)$$

### 5. Convergence to BS-type models

In this section we will illustrate the convergence of the binomial and trinomial option pricing models. More particularly, we will examine the plain vanilla European call and American put, both specified in three subsequent forms: as ATM, ITM, and OTM options. The time to maturity is one year, $\tau = 1$, the underlying asset is of the non-dividend type with initial price $S_0 = 100$, the volatility of its returns $\sigma = 25\%$ p.a., and the riskless rate $r = 5\%$. We will apply the binomial model of the CRR2 form (see Table 1) and the trinomial model with parameters as specified in section 4.

Suppose first the vanilla call, initially at-the-money ($K_{ATM} = S = 100$). The specification of the binomial model parameters implies, that the strike price either matches the middle node $S_T = S_0 u^{n/2} d^{n/2} = S_0$ (the number of steps is even) or the strike price represent such imaginary node (the number of steps is odd).

![Figure 1: Convergence of Binomial and Trinomial models for European call option initially set as ATM, OTM, ITM (left to right) ](image)

According to Figure 1, we can see that if we set the number of steps to be even, we underprice the option, while for the odd number of steps we overprice the option. As we increase the number of steps, the resulting value will oscillate around the Black-Scholes...
value. By contrast, since trinomial model allows us to get always back to the initial value \( S_0 \), it will always converge to the true result from below.

In order to examine the convergence of non-at-the-money options, we have set also \( K_{ITM} = 105 \) and \( K_{OTM} = 95 \). The results are obvious from Figure 1. We can see, that the binomial model still oscillate, starting from, say, \( n = 10 \). However, it does not oscillate around any constant line. Sometimes, in average, it overprices the option, sometimes, in average, it underprices the option. Note, that there can be calculated such \( n \) for which we arise exactly at the (theoretically) true result – we must match the strike price by any of the nodes.

Although the trinomial model oscillate too, its convergence is much better. When it breaks the Black-Scholes line for the second time, it keeps very close by it.

Since the application of the lattice models lies mainly in valuation of American options, and the American call is equivalent to the European call (with no dividend-type payment), we will examine the convergence on the case of put options. We set the strike prices as before, \( K_{ATM} = 100 \), \( K_{ITM} = 105 \), and \( K_{OTM} = 95 \). The results are depicted at Figure 2.

![Figure 2: Convergence of Binomial and Trinomial models for American put option initially set as ATM, OTM, and ITM (left to right)](image)

We can see that the Black and Scholes model highly underprices the options, for about 10%. The convergence does not seem to be exactly the same as in the case of European call. Start with the ATM option. Although the binomial model still oscillate around some constant line (possible the fair value of the option), the results obtained by application of trinomial model are slightly lower.

Similarly, the shape of the convergence curve of the binomial model for the ITM and OTM options is similar to European call (even if not so much floating). By contrast, the fluctuation of the trinomial model seems to have totally different manner.

6. Conclusions

In this paper we have presented two basic forms of lattice models of option valuation – the binomial and trinomial approaches. We have described the derivation of both models. Next, we have examined the convergence of these models on the case of European plain vanilla call and American put option.

Although there can be some effect of particular parameters, the essentials of the convergence seem to be valid generally. In order to get better results, e.g. when pricing exotic options, more complex and advanced approaches, such as adaptive mesh models or...
models matching the strike price/barrier level, should be used. However, the basic models presented here indicate us the usefulness for real options application.

References


\[ f(3,3) = \Psi[S(3,3)] \]
\[ f(3,1) = \Psi[S(3,1)] \]
\[ f(3,-1) = \Psi[S(3,-1)] \]
\[ f(3,-3) = \Psi[S(3,-3)] \]

Figure 3: Binomial model for more periods (CRR2; \(\Delta t = 1\), \(n = 3\))

\[ S(3,3) = S_0 uuu \]
\[ S(3,2) = S_0 uu \]
\[ S(3,1) = S_0 u \]
\[ S(3,0) = S_0 \]
\[ S(3,-1) = S_0 d \]
\[ S(3,-2) = S_0 dd \]
\[ S(3,-3) = S_0 ddd \]

Figure 4: Trinomial model for more periods
Summary
Konvergence binomického a trinomického modelu oceňování opcí